

# Absolute Continuity in Partial Differential Equations

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## Abstract

In this note we study a function which frequently appears in partial differential equations. We prove that this function is absolutely continuous, hence it can be written as a definite integral. As a result we obtain some estimates regarding solutions of the Hamilton-Jacobi systems.

*Key Words:* Absolute continuity, Weak convergence, Rearrangements of functions, Measure preserving maps, Hamilton-Jacobi systems, Estimates.

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## 1 Introduction

Let  $H$  be a differential operator of order  $m \in \mathbb{N}$  and let  $f \in L^p(D)$  be a positive function, where  $p \in (1, \infty)$  and  $D$  is a smooth bounded domain in  $\mathbb{R}^n$ . Consider the equation:

$$H(u) = f, \quad \text{in } D \tag{1}$$

A function  $u \in W^{m,p}(D) \cap C(\overline{D})$  is called a strong solution of (1) provided that  $H(u) = f$  almost everywhere (a. e.) in  $D$ . We assume the operator  $H$  satisfies the following condition:

$$\text{For any } u \in W^m(D) \text{ and } \gamma \in \mathbb{R} : H(u) = 0 \text{ a. e. in } E_\gamma := \{x \in D \mid u(x) = \gamma\} \tag{P}$$

For a measurable function  $h : D \rightarrow \mathbb{R}$ , the distribution function of  $h$ , denoted  $\lambda_h(\alpha)$ , is defined as follows:

$$\lambda_h(\alpha) := |\{x \in D \mid h(x) \geq \alpha\}| \equiv |\{h \geq \alpha\}|, \quad (\forall \alpha \in \mathbb{R})$$

where  $|\cdot|$  denotes the  $n$ -dimensional Lebesgue measure. Clearly  $\lambda_h$  is decreasing, and if  $h$  is continuous, then  $\lambda_h$  will be strictly decreasing. Moreover, in case the graph of  $h$  has no significant flat sections

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(i. e.  $\forall \gamma \in \mathbb{R} : |\{h = \gamma\}| = 0$ ) then  $\lambda_h$  will be continuous. The decreasing rearrangement of  $h$ , denoted  $h^*(s)$ , is defined as follows:

$$\begin{cases} h^* : [0, |D|] \rightarrow \mathbb{R} \\ h^*(s) = \inf\{\alpha \mid \lambda_h(\alpha) \leq s\} \end{cases}$$

Note that when  $h$  is continuous and its graph has no significant flat sections then:

$$\lambda_h \circ h^*(s) = s \quad \text{and} \quad h^* \circ \lambda_h(\alpha) = \alpha.$$

We also need to recall some background from rearrangements of functions. Given  $g_0 : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ , the rearrangement class generated by  $g_0$ , denoted  $\mathcal{R}(g_0)$ , is the set of functions  $g : D \rightarrow \mathbb{R}$  such that  $\lambda_g(\alpha) = \lambda_{g_0}(\alpha)$ , for every real  $\alpha$ . In case  $g_0 \in L^p(D)$  then  $\mathcal{R}(g_0) \subseteq L^p(D)$ , and  $\forall g \in \mathcal{R}(g_0) : \|g\|_p = \|g_0\|_p$ . The weak closure of  $\mathcal{R}(g_0)$  in  $L^p(D)$  is denoted as  $\overline{\mathcal{R}(g_0)}$  which, unlike  $\mathcal{R}(g_0)$ , enjoys some nice properties and characterizations that are stated in the following lemma. For the proof and further reading see [Bur87, Bur89, ET87, Kaw85]:

**Lemma 1.1.** *Let  $g_0 \in L^p(D)$  be a non-negative function, and  $\mathcal{R}(g_0)$  be the rearrangement class generated by  $g_0$ . Then:*

- (1)  $\overline{\mathcal{R}(g_0)}$  is convex, and weakly compact in  $L^p(D)$ .
- (2)  $\overline{\mathcal{R}(g_0)} = \overline{\text{co}(\mathcal{R}(g_0))}$ , the closed convex hull of  $\mathcal{R}(g_0)$ .
- (3) The following characterization stands:

$$\overline{\mathcal{R}(g_0)} = \left\{ g \mid \forall s \in (0, |D|) : \int_0^s g^*(t) dt \leq \int_0^s g_0^*(t) dt, \text{ and } \int_0^{|D|} g^*(t) dt = \int_0^{|D|} g_0^*(t) dt \right\}$$

The set of measure-preserving maps from  $D$  onto  $[0, |D|]$  is a non-empty set (e. g. see [Roy63, Chapter 11]) which will be denoted by  $\mathcal{M}(D, [0, |D|])$ . By a result attributed to Ryff [Ryf70], given  $g : D \rightarrow \mathbb{R}$ , there exists  $\phi \in \mathcal{M}(D, [0, |D|])$  such that  $g = g^* \circ \phi$  almost everywhere in  $D$ .

We now introduce the function that is the main drive behind writing this note. To this end, we assume  $u \in W^{m,p}(D) \cap C(\overline{D})$  is a strong solution of (1). We are interested in the function  $\xi : [0, |D|] \rightarrow \mathbb{R}$  defined by:

$$\xi(s) = \int_{\{u \geq u^*(s)\}} f(x) dx. \quad (2)$$

Thanks to property **(P)** on the preceding page, and of course the fact that  $f$  is positive, the level sets  $\{u = \gamma\}$  must have zero measure, hence  $\xi$  is well-defined. This function is frequently referred to in partial differential equations, particularly when one is interested in comparing the solution of a boundary value problem to that of a symmetrized problem, the latter being readily solved. There are many references in this regard, e. g. [BFK03, GN84, Tal76], to mention a few. In this note we prove that  $\xi$  is absolutely continuous, hence it can be represented by a definite integral of the form  $\int_0^s F(\tau) d\tau$ . Then, we will prove that the integrand  $F$  composed with any measure-preserving map  $\phi \in \mathcal{M}(D, [0, |D|])$  belongs to  $\overline{\mathcal{R}(f)}$ . Using these two results we point out a couple of applications.

Throughout this paper we use some standard notations. For example,  $W^{m,p}(D)$  and  $W^m(D)$  denote the usual Sobolev spaces. The space  $L^p(D)$  comprises functions whose  $p$ -th powers are integrable, and the norm in this space is defined by  $\|f\|_p = \left(\int_D |f|^p dx\right)^{1/p}$ . Moreover,  $C(D)$  and  $C(\bar{D})$  denote the spaces of continuous functions over  $D$  and its closure  $\bar{D}$ , respectively, and the corresponding norm is denoted by  $\|\cdot\|_\infty$ . The arrow “ $\rightarrow$ ” indicates strong convergence, whilst “ $\rightharpoonup$ ” indicates weak convergence in spaces under discussion.

## 2 Main results

Our first main result is the following:

**Theorem 2.1.** *The function  $\xi$ , as defined in (2), is absolutely continuous on  $[0, |D|]$ .*

*Proof.* Let  $\epsilon > 0$ , and consider a finite sequence  $\{(\alpha_i, \beta_i) \mid 1 \leq i \leq N\}$  of non-overlapping subintervals of  $[0, |D|]$  such that  $\sum_{i=1}^N (\beta_i - \alpha_i) < \delta$ , where  $\delta$  is a positive number to be determined later. By setting  $t(\alpha_i) = u^*(\alpha_i)$  and  $t(\beta_i) = u^*(\beta_i)$  we will have:

$$\sum_{i=1}^N |\xi(\beta_i) - \xi(\alpha_i)| = \sum_{i=1}^N \left| \int_{\{t(\beta_i) < u < t(\alpha_i)\}} f(x) dx \right| = \int_E f(x) dx, \quad (3)$$

where  $E = \cup_{i=1}^N \{x : u^*(\beta_i) < u(x) < u^*(\alpha_i)\}$ . By applying the Hölder inequality we obtain:

$$\int_E f(x) dx \leq |E|^{\frac{1}{q}} \|f\|_p, \quad (4)$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ . Note that  $|E| = \sum_{i=1}^N (\beta_i - \alpha_i)$ . This, along with (3) and (4), will give the desired result, provided that  $\delta < \left(\frac{\epsilon}{\|f\|_p}\right)^q$ .  $\square$

**Corollary 2.2.** *The function  $\xi$ , as defined in (2), satisfies:*

$$\xi(s) = \int_0^s F(\tau) d\tau, \quad (5)$$

for some integrable function  $F$ .

*Proof.* By Theorem 2.1,  $\xi$  is absolutely continuous. Hence we can apply Corollary 14 in [Roy63], together with the fact that  $\xi(0) = 0$ , to deduce

$$\xi(s) = \int_0^s \xi'(\tau) d\tau,$$

almost everywhere in  $[0, |D|]$ . So by setting  $F(s) = \xi'(s)$ , we get the desired result.  $\square$

We now state our second main result:

**Theorem 2.3.** *Let  $F$  be the function in Corollary 2.2 and  $\phi \in \mathcal{M}(D, [0, |D|])$ . Then  $F \circ \phi \in \overline{\mathcal{R}(f)}$ .*

*Proof.* Note that  $\lambda_{F \circ \phi}(\alpha) = \lambda_F(\alpha)$ , for every  $\alpha \in \mathbb{R}$ . Thus,  $(F \circ \phi)^*(s) = F^*(s)$ , for almost every  $s \in [0, |D|]$ . Hence, in view of item (3) of Lemma 1.1, it suffices to prove:

- (i)  $\int_0^{|D|} F^*(s) ds = \int_0^{|D|} f^*(s) ds.$
- (ii)  $\int_0^s F^*(t) dt \leq \int_0^s f^*(t) dt, \quad \forall s \in (0, |D|).$

Proving (i) is straightforward as

$$\begin{aligned} \int_0^{|D|} F^*(t) dt &= \int_0^{|D|} F(t) dt = \xi(|D|) \\ &= \int_{\{u \geq t(|D|)\}} f dx = \int_{\{u \geq 0\}} f dx = \int_D f dx = \int_0^{|D|} f^*(t) dt, \end{aligned}$$

where we have used Corollary 2.2.

To prove (ii), we consider the following steps:

*Step 1.* Let  $\mathcal{U}$  be an open subset of  $(0, |D|)$ . Then, we can write  $\mathcal{U} = \cup_{i=1}^{\infty} (A_i, B_i)$ , where  $(A_i, B_i)$  are mutually disjoint. Hence,

$$\begin{aligned} \int_{\mathcal{U}} F(\tau) d\tau &= \sum_{i=1}^{\infty} \int_{A_i}^{B_i} F(\tau) d\tau = \sum_{i=1}^{\infty} \left( \int_0^{B_i} F(\tau) d\tau - \int_0^{A_i} F(\tau) d\tau \right) \\ &= \sum_{i=1}^{\infty} \left( \int_{\{u \geq t(B_i)\}} f dx - \int_{\{u \geq t(A_i)\}} f dx \right) = \sum_{i=1}^{\infty} \int_{\{t(B_i) \leq u < t(A_i)\}} f dx \\ &= \int_{\cup_{i=1}^{\infty} \{t(B_i) \leq u < t(A_i)\}} f dx \leq \int_0^{|\cup_{i=1}^{\infty} \{t(B_i) \leq u < t(A_i)\}|} f^*(s) ds \\ &= \int_0^{\sum (B_i - A_i)} f^*(s) ds = \int_0^{|\mathcal{U}|} f^*(s) ds. \end{aligned}$$

*Step 2.* Let  $\mathcal{V}$  be a measurable subset of  $(0, |D|)$ , and  $\epsilon > 0$ . By Theorem 3.6 in [WZ77], there exists an open set  $G$  containing  $\mathcal{V}$  such that  $|G \setminus \mathcal{V}| < \epsilon$ . Whence

$$\begin{aligned} \int_{\mathcal{V}} F(t) dt &\leq \int_G F(t) dt \leq \int_0^{|G|} f^*(s) ds \\ &= \int_0^{|\mathcal{V}|} f^*(s) ds + \int_{|\mathcal{V}|}^{|G|} f^*(s) ds \\ &\leq \int_0^{|\mathcal{V}|} f^*(s) ds + \|f\|_p (|G| - |\mathcal{V}|)^{1/q}, \end{aligned} \tag{6}$$

where we have used Step 1, and Hölder's inequality. Since  $|G| - |\mathcal{V}| = |G \setminus \mathcal{V}| < \epsilon$ , from (6) we infer

$$\int_{\mathcal{V}} F(t) dt \leq \int_0^{|\mathcal{V}|} f^*(s) ds + \epsilon^{1/q} \|f\|_p. \tag{7}$$

Since  $\epsilon$  is arbitrary, (7) implies

$$\int_{\mathcal{V}} F(t) dt \leq \int_0^{|\mathcal{V}|} f^*(s) ds.$$

*Step 3.* We recall the following maximization from [ATL89]:

$$\sup_{\{\omega \subseteq [0, |D|] : |\omega| = \gamma\}} \int_{\omega} F(t) dt = \int_0^{|\omega|} F^*(s) ds.$$

Now, fix  $s \in (0, |D|)$ , and apply Step 3 to obtain

$$\sup_{\{\omega \subseteq [0, |D|] : |\omega| = s\}} \int_{\omega} F(t) dt = \int_0^s F^*(t) dt. \quad (8)$$

On the other hand, from Step 2, we have:

$$\int_{\omega} F(t) dt \leq \int_0^{|\omega|} f^*(s) ds. \quad (9)$$

From (8) and (9) we deduce

$$\int_0^s F^*(t) dt \leq \int_0^s f^*(t) dt,$$

as desired.  $\square$

**Corollary 2.4.** *Suppose the hypotheses of Theorem 2.3 hold. Then, there exists a sequence of functions  $\{F_n\}$  such that  $F_n^*(s) = f^*(s)$  and  $F_n \rightarrow F$  in  $L^p(0, |D|)$ .*

*Proof.* By Ryff's result,  $f = f^* \circ \phi$ , for some  $\phi \in \mathcal{M}(D, [0, |D|])$ . From Theorem 2.3, we infer  $F \circ \phi \in \overline{\mathcal{R}(f)}$ . So, there exists a sequence  $\{f_n\} \subseteq \mathcal{R}(f)$  such that  $f_n \rightarrow F \circ \phi$  in  $L^p(D)$ . Therefore,  $f_n \circ \phi^{-1} \rightarrow F$  in  $L^p(0, |D|)$ . Clearly,  $\lambda_{f_n \circ \phi^{-1}}(\alpha) = \lambda_f(\alpha)$ , so  $(f_n \circ \phi^{-1})^*(s) = f^*(s)$ . This completes the proof.  $\square$

### 3 Applications

In this section we will present a couple of applications of the results of the previous section. Throughout we will assume the extra condition  $f \in C(\overline{D})$ . Let us consider the following Hamilton-Jacobi system:

$$\begin{cases} |\nabla u| = f(x), & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases} \quad (10)$$

**Lemma 3.1.** *The system (10) has a strong positive solution  $u \in W^{1,\infty}(D)$ .*

*Proof.* From [Lio82] we know that the system (10) has a strong solution  $u \in W^{1,\infty}(D)$ . Replacing  $u$  by  $|u| \in W^{1,\infty}(D)$  if necessary, taking into account that  $|\nabla(|u|)| = |\nabla u|$ , we can assume  $u$  is non-negative. On the other hand, since  $f$  is positive, we can apply Lemma 7.7 in [GT01], to ensure that the level sets  $\{u = \gamma\}$  have zero measure. Thus,  $u$  is essentially positive, as desired.  $\square$

*Remark 3.1.* For  $f$  and  $u$  as in Lemma 3.1, the function:

$$\xi(s) = \int_{\{u \geq s\}} f(x) dx, \quad (\text{where } s = \lambda_u(t)),$$

is well defined. As a result, the function  $F$  from Corollary 2.2 is also well defined. Moreover, the conclusions of Theorem 2.1 and Theorem 2.3 hold.

Our first application is as follows:

**Theorem 3.2.** *Let  $u \in W^{1,\infty}(D)$  be a strong positive solution of the Hamilton-Jacobi system (10) and let  $v$  be the unique solution of the following system:*

$$\begin{cases} |\nabla Z| = F(\omega_n |x|^n), & \text{in } B \\ Z = 0, & \text{on } \partial B, \end{cases} \quad (11)$$

in which:

- $B$  is the ball centred at the origin with radius  $(|D|/\omega_n)^{1/n}$ , and  $\omega_n$  indicates the volume of the unit  $n$ -dimensional ball.
- The function  $F$  is as in Corollary 2.2, which is well defined by Remark 3.1.

Also, let  $u^\sharp(x) \equiv u^*(\omega_n |x|^n)$ , which in the literature is referred to as the Schwarz symmetrization of  $u$ . Then,  $u^\sharp(x) \leq v(x)$ , for  $x \in B$ .

*Proof.* The proof is a consequence of Corollary 2.2, along the same lines as in the proof of Lemma 2.2 in [GN84].  $\square$

*Example 3.1.* Choosing  $f(x) = 1$  in Theorem 3.2 yields  $F(t) = 1$ . Thus, the conclusion of Theorem 3.2 states:

$$u^\sharp(x) \leq v(x) = R - |x|, \quad x \in B,$$

where  $R = (|D|/\omega_n)^{1/n}$ . This estimate can be obtained directly as follows:

$$\begin{aligned} \lambda_u(t) &= \int_{\{u \geq t\}} dx = \int_{\{u \geq t\}} |\nabla u| dx \\ &= \int_t^{\|u\|_\infty} \left( \int_{\{u=\tau\}} dH^{n-1} \right) d\tau = \int_t^{\|u\|_\infty} P(\{u \geq \tau\}) d\tau, \end{aligned} \quad (12)$$

where we have used the co-area formula (e. g. see [Mor09]). Here,  $P(E)$  stands for the perimeter of  $E$  in the sense of De Giorgi. By differentiating (12), and applying the classical Isoperimetric Inequality (e. g. see [Hen06]), we derive:

$$\lambda'_u(t) = -P(\{u \geq t\}) \leq -n\omega_n^{\frac{1}{n}} \lambda_u^{1-\frac{1}{n}}(t).$$

Thus, we obtain:

$$1 \leq -\frac{\lambda'_u(t)}{n\omega_n^{1/n}\lambda_u^{1-\frac{1}{n}}(t)}. \quad (13)$$

Integrating (13) from 0 to  $t$  leads to:

$$\begin{aligned} t &\leq -\frac{1}{n\omega_n^{1/n}} \int_0^t \frac{\lambda'_u(\tau)}{\lambda_u^{1-\frac{1}{n}}(\tau)} d\tau = -\frac{1}{n\omega_n^{1/n}} \int_{|D|}^{\lambda_u(t)} \frac{ds}{s^{1-\frac{1}{n}}} \\ &= \frac{1}{\omega_n^{1/n}} (|D|^{1/n} - \lambda_u^{1/n}(t)) = R - \left(\frac{\lambda_u(t)}{\omega_n}\right)^{1/n}. \end{aligned} \quad (14)$$

By letting  $t = u^*(\omega_n|x|^n)$  in (14), and recalling  $\lambda_u(u^*(\omega_n|x|^n)) = \omega_n|x|^n$ , we obtain  $u^\sharp(x) \leq R - |x|$  for  $x \in B$ , as expected.

The second application is stated in the following Theorem:

**Theorem 3.3.** *Let  $u$  be as in Theorem 3.2. Then*

$$\|u\|_\infty \leq C|D|^{1/n}\|f\|_\infty.$$

*Proof.* The proof is a consequence of Corollary 2.4, along the same lines as in the proof of Corollary 2.1 in [GN84].  $\square$

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