Rearrangement Optimization Problems: A Computational Perspective

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Outline

Historical Background

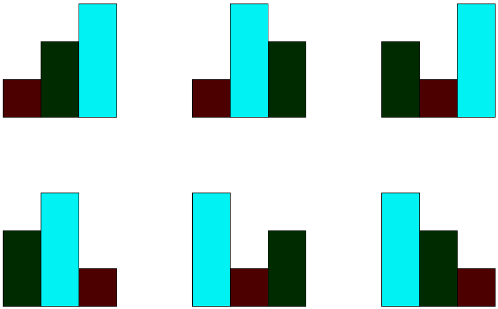
- Rearrangements can be traced back to 1899 in hydrostatics:
  - Carlo Somigliana (1860–1935), the University of Turin (Italy).
  - See (Talenti 2016) for a survey of rearrangements.
- Rearrangement Optimization Problems:
  - Systematically studied by Burton (1987).
  - In the context of fluid dynamics (vortex rings)

  - Based on a formulation by Benjamin (1976).

Historical Background

- Since Burton's work, ROPs have been studied in other areas, such as:
  - Construction of Robust Membranes (Emamizadeh, Farjudian, Liu, and Marras (2017))
  - Finance (Emamizadeh and Hanai (2009))
  - Free Boundary Problems (Emamizadeh and Marras (2014))
  - Marine Economy (Emamizadeh, Farjudian, and Liu (2017)).
  - ...

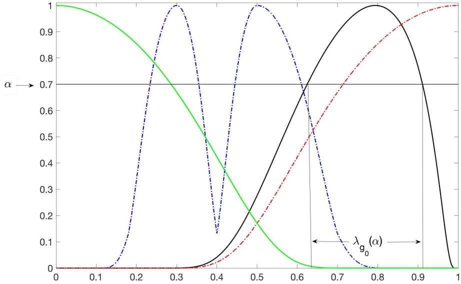
Discrete Rearrangement: Permutation



- In the discrete case, rearrangement becomes permutation.
- Thus, for n items, there are at most $n!$ rearrangements.

Continuous Rearrangement

- $g_0 : [0, 1] \to \mathbb{R}$ (in **thick black**), and three of its rearrangements



- Distribution function: $\lambda_g(\alpha) := |\{x \in [0, 1] \mid g(x) \geq \alpha\}|$.
- g is a rearrangement of g_0 iff $\forall \alpha \geq 0 : \lambda_{g_0}(\alpha) = \lambda_g(\alpha)$.

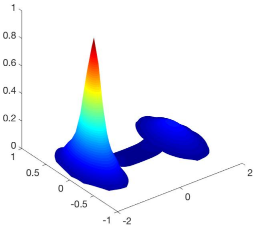
Continuous Rearrangement

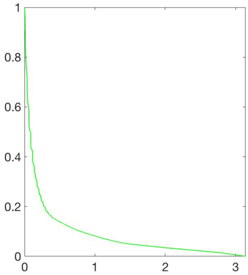
- In general, the domains can be different.
- Let (Ω_1, M_1, μ_1) and (Ω_2, M_2, μ_2) be two measure spaces such that $\mu_1(\Omega_1) = \mu_2(\Omega_2)$.
- Two *measurable* functions $f : \Omega_1 \to \mathbb{R}$ and $g : \Omega_2 \to \mathbb{R}$ are said to be rearrangements of each other, denoted as $f \simeq g$, iff:

$$\forall \alpha \in \mathbb{R} : \mu_1(f^{-1}[\alpha, \infty)) = \mu_2(g^{-1}[\alpha, \infty))$$

Continuous Rearrangement

- A function f on a dumbbell shaped domain (left), together with its decreasing rearrangement (right).





- The size of the dumbbell shaped domain is π.

Rearrangement Class

In this talk, we usually have $\Omega_1 = \Omega_2$:

Definition (Rearrangement Class $\mathcal{R}(g_0)$)

The rearrangement class $\mathcal{R}(g_0)$ ***generated*** by $g_0 : \Omega \to \mathbb{R}$ is defined as:

$$\mathcal{R}(g_0) := \{g : \Omega \to \mathbb{R} \mid g \text{ is a rearrangement of } g_0\}.$$

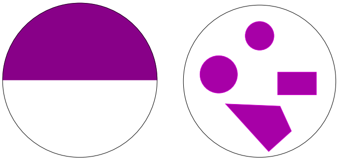
Outline

An Abstract Formulation of ROPs

- **Informal**: Does there exist an optimal rearrangement of g_0 for the problem at hand?
- **More formal**: For a given generator $g_0 \in L^p(\Omega)$, and (energy) functional $\Phi : L^p(\Omega) \to \mathbb{R}$, is there an element of $\mathcal{R}(g_0)$ which optimizes Φ?
- Note:
  - In the finite (discrete) case, the answer is always **'yes'**.
  - In the continuous case, even the very existence of solutions is not always guaranteed (See, e. g., (Liu, Emamizadeh, and Farjudian 2016) for a natural example).

Example: Constructing a Robust Membrane

- Goal:
  - constructing a clamped membrane,
  - which is subject to a vertical force $f(x)$ at each point x,
  - such that the membrane is as robust as possible.
- Input:
  - n materials of densities $\alpha_1, \alpha_2, \ldots, \alpha_n$;
  - These cannot be altered, e. g., reduced, increased, mixed, etc.
- Example: Disc shaped domain, and 2 materials
  - Which arrangement is best? (higher density in magenta)



. . . . . . any other?

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Robust Membrane: ROP Form

- For a given
  - domain $\Omega \subseteq \mathbb{R}^d$,
  - non-negative force function $f \in L^2(\Omega)$,
  - generator $g_0 \in L^\infty(\Omega)$,
- is there a rearrangement $g \in \mathcal{R}(g_0)$ which minimizes:

$$\Phi(g) := \int_D f u_g dx,$$

- in which u_g is the unique solution of:

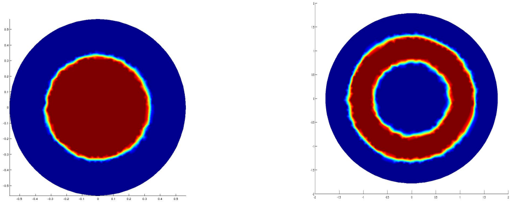
$$\begin{cases} -\Delta u + g(x)u = f(x) & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases}$$

$\mathcal{R}$ and $\overline{\mathcal{R}}$

- The main obstacle is that the rearrangement class $\mathcal{R}$ does not have the usual favorable properties for optimization, i. e.
  - it is neither convex;
  - nor compact.
- The natural candidate would be the weak closure $\overline{\mathcal{R}}$ in $L^2(\Omega)$, which turns out to be:
  - convex and compact;
  - and, to contain $\mathcal{R}$ as the set of its extremal points.
- **General procedure**:
  - Relax the original problem from $\mathcal{R}$ to $\overline{\mathcal{R}}$;
  - Show that it is solvable over $\overline{\mathcal{R}}$.
  - Then, use a suitable method to prove that a solution does indeed lie in $\mathcal{R}$.

Back to Robust Membranes

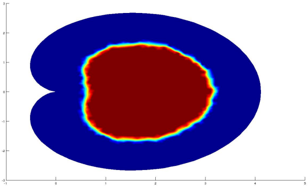
- The robust membrane problem does indeed have a solution in the rearrangement class (Emamizadeh, Farjudian, Liu, and Marras (2017)).
- Disc shaped domain. $f(x) := 10$ (left) and $\forall x \in D : f(x) := \frac{1}{0.01+\delta(x)^2}$ (right), in which $\delta(x)$ is the distance of each point x to the boundary of the domain Ω:

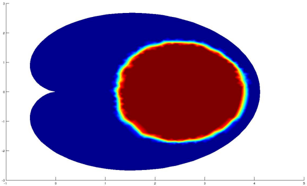


(High density material in red, low density in blue)

Robust Membranes: Cardioid

- Cardioid domain: $f_1(x, y) := 5$ (left), and $f_2(x, y) := 5 + x^3$ (right).

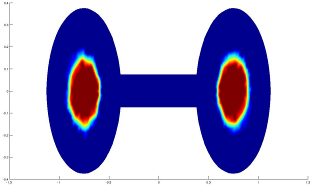




- Note how the higher density material (red) should be used where it matters.

Robust Membranes: Symmetry Preservation

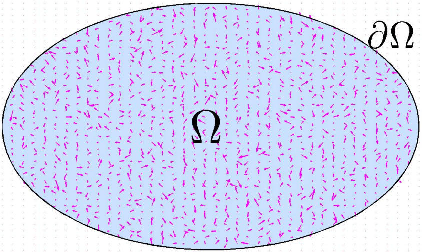
- Dumbbell shaped domain: $f(x,y) := 1$:



- Notice the preservation of Steiner symmetry.

Example: Optimal Harvesting of a Marine Species

- A population of fish in a lake.
- Source of income.
- Sustainability.
- Challenge: balance between harvesting and well-being of the species.

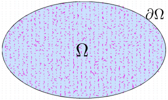


Mathematical modelling (Reaction-Diffusion)

- Dynamics of the population density $u : \Omega \times \mathbb{R} \to \mathbb{R}$

$$
\begin{cases}
\underbrace{\dfrac{\partial u}{\partial t}}_{\text{rate of change}} = \underbrace{\epsilon^2 \Delta u}_{\text{diffusion}} + \overbrace{au - bu^2}^{\textit{reaction}} - \underbrace{E(x)u}_{\text{harvesting}} & (x \in \Omega, t \in \mathbb{R}) \\
u(x,t) = 0 & (x \in \partial\Omega, t \in \mathbb{R}) \\
u(x,0) = u_0(x) & (x \in \Omega)
\end{cases}
$$

- $E(x)$ is *harvesting effort*, the only parameter we can control.



Assumptions (Kurata and Shi (2008))

- Harvesting effort is bounded:

$$\exists M > 0, \forall x \in \Omega : 0 \leq E(x) \leq M$$

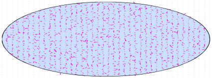
- Among all the admissible choices for $E(x)$, the total effort is constant:

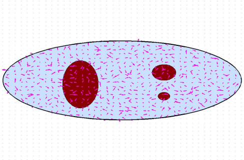
$$\exists \beta > 0 : \int_\Omega E(x)\,\mathrm{d}x = \beta|\Omega|,$$

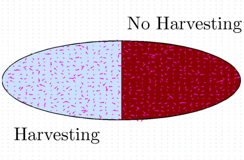
in which $\beta > 0$ is the average effort.

Problem

What is the best harvesting strategy $E(x)$ that *minimizes* harm to the species?







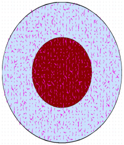
Optimization of energy functional

- Let u_E be the solution of the following BVP for a given strategy E:
$$\begin{cases} \epsilon^2 \Delta u + u - u^2 - E(x)u = 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$
- To minimize harm to the species we need to find a strategy $E(x)$ for which the following energy is minimum:

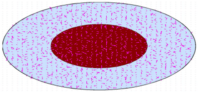
$$\Phi(E) := \underbrace{\frac{\epsilon^2}{2} \int_\Omega |\nabla u_E|^2 \, \mathrm{d}x}_{\text{kinetic energy}} - \underbrace{\frac{1}{2} \int_\Omega u_E^2 \, \mathrm{d}x + \frac{1}{3} \int_\Omega u_E^3 \, \mathrm{d}x + \frac{1}{2} \int_\Omega E(x) u_E^2 \, \mathrm{d}x}_{\text{potential energy}}$$

Symmetry preserving of the best strategy

- ***It turns out that the domain must always be partitioned into a no harvesting zone and a maximum harvesting zone.***
- If Ω is a disc, then no harvesting in the center:

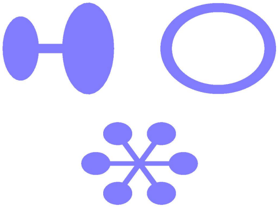


- If Ω is an ellipse, the same:



Other Domains

- What if the domain is not convex?



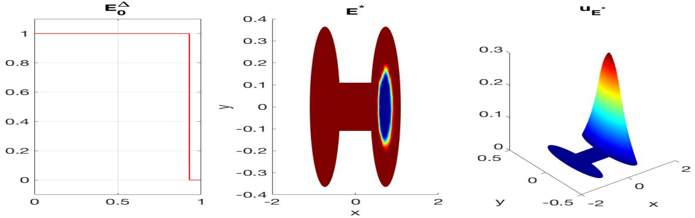
- Analytic methods cannot give us the exact location for every case.
- We need *numerical algorithms*.

Optimal Harvesting as a ROP (Emamizadeh, Farjudian, and Liu (2017))

- When the generator is a *characteristic function*, we get the result of Kurata and Shi (2008).
- For ***arbitrary*** generators—i. e., when the boats in the fleet have various capacities—Emamizadeh, Farjudian, and Liu (2017) prove:
  - Existence of solutions.
  - Preservation of Steiner symmetry.
- and provide a fast numerical method.

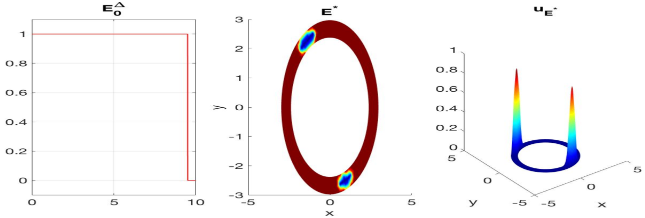
Symmetry Breaking (Dumbbell)

- Steiner symmetry is preserved, but symmetry w. r. t. the y axis is not.



(Left to right: non-increasing rearrangement of the generator E_0, the optimal harvesting strategy E^*, the population density for the optimal strategy)

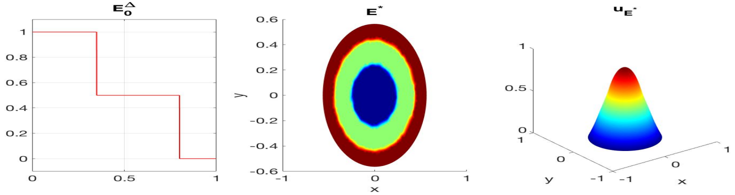
Symmetry Breaking (Annulus)



(Left to right: non-increasing rearrangement of the generator E_0, the optimal harvesting strategy E^*, the population density for the optimal strategy)

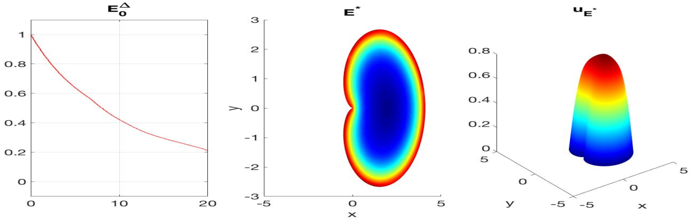
Generator taking 3 Values.

- More aggressive harvesting should be near the boundary (red).
- No harvesting in the center (blue).
- Mid-range boats should be deployed in between (green).



Generator with No Flat Sections.

- Again, the more aggressive harvesting should be carried out near the boundary:



Outline

1. Introduction
2. Rearrangement Optimization Problems
3. Computational Aspects

Computability

Question

For a given ROP, what are the conditions under which the solution(s) is/are computable?

- Computability is to be taken in the sense of Type-II Theory of Effectivity (Weihrauch 2000).
- The answer in the general case is not known yet.

Stability

- Floating-point computations have inherent inaccuracies.
- How numerically stable are our algorithms for solving ROPs?
- We discuss two specific questions:

Question

If a sequence of generators (f_n) converges to f in an appropriate L^p space, does the sequence of optimizers $(\hat{f}_n)$ also converge to $\hat{f}$ in the same space?

Question

If f_n converges to f in an appropriate L^p space, does the Hausdorff distance between the weak closures $\overline{\mathcal{R}(f_n)}$ and $\overline{\mathcal{R}(f)}$ also tend to zero?

Stability Theorems

- We have proven the answers to both of the questions to be ‘yes’. (Liu, Emamizadeh, and Farjudian (2016)).
- Nonetheless, these are just two components of a complicated numerical procedure.

Complexity

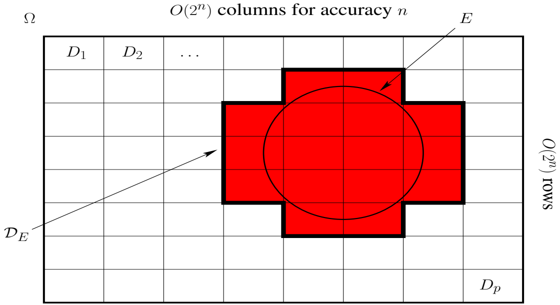
- Rearrangement classes are subsets of infinite-dimensional Banach spaces.
- Numerical methods work on finite dimensional subsets, mostly based on finite element methods.
- Even for modestly refined meshes, the size of the search space becomes very large.

Complexity: 2D Shape Optimization

- Let us consider the simple case of $\Omega \subseteq \mathbb{R}^2$,
- and generator $g_0 = \chi_{E_0}$, where $E_0 \subseteq \Omega$.
  - χ_{E_0} is the characteristic function of E_0.
- Suppose that we have a mesh $\mathcal{D} = \{D_i \mid 1 \leq i \leq P\}$ (for some $P \in \mathbb{N}$) over the domain $\Omega \subseteq \mathbb{R}^2$ such that $\Omega = \cup\mathcal{D}$.

Discretization

- Let $\mathcal{D}_E = \{D_{i_j} \mid 1 \leq i_j \leq P\}$ be the best outer approximation of the set $E \subseteq \Omega$, i. e. $\mathcal{D}_E$ is a subgrid of $\mathcal{D}$ with the smallest area for which $E \subseteq \cup\mathcal{D}_E$.
- We define $\#E$ as the number of indices k such that $D_k \in \mathcal{D}_E$.



Complexity: 2D Shape Optimization

- Using Stirling's formula, for the search space $\mathbb{S}$ we get:

$$|\mathbb{S}| \approx \binom{P}{\# E_0} \geq C \frac{\zeta^{2^{2n}}}{2^n}$$

- where:
  - $1 < \zeta \leq 2$ depends on $\rho := \frac{|E_0|}{|\Omega|}$;
  - C depends on ρ and Ω;
  - n denotes the required accuracy (say, the number of correct digits of the result).
- For instance: for $\rho := 0.15$, and (very modest) accuracy $n := 5$: $\zeta^{2^{2n}}/2^n \geq 3 \times 10^{186}$.
- For $\rho = 0.5$, and $n = 5$, $\zeta^{2^{2n}}/2^n \geq 5 \times 10^{306}$.

Intractability

- Thus, in general, ROPs are intractable problems.
- In practice, there are sometimes ways of speeding up the computation process.
  1. Using a very fast method by Eydeland and Turkington (1988), when applicable. (provides geometric rate of convergence)
  2. Failing that, using the derivative of the energy functional, if it exists. (quite slower, but feasible)
- Nonetheless, there are no solid proofs that these methods *always* get us to the correct answer, even when they are applicable.
- The main problem lies in the presence of local optima and saddle points.

Randomization

- In cases where there are (uncountably many) local optima and saddle points present, randomization is needed to:
  - Escape local optima and saddle points.
  - Avoid brute force search.
- Examples:
  - Optimal harvesting, with appropriate parameters, over non-convex domains (Emamizadeh, Farjudian, and Liu 2017).
  - Nonlocal problems of Kirchhoff type (Emamizadeh, Farjudian, and Zivari-Rezapour 2016).

Summary

- ROPs have a simple abstract formulation.
- As such, they turn up in diverse areas of mathematics and applied sciences.
- There is a great scope for research on computational aspects of ROPs:
  - Computable analysis of ROPs.
  - Stability.
  - Complexity analysis.
  - Connections to discrete computability and complexity.
  - Novel numerical methods for rearrangement optimization problems.
  - Developing interval libraries for ROPs (especially important for the study of symmetry breaking).
  - Finding applications of ROPs in computer science.

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