Optimization problems with fixed volume constraints and stability results related to rearrangement classes

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Abstract

The material in this paper has been divided into two main parts. In the first part we describe two optimization problems—one maximization and one minimization—related to a sharp trace inequality that was recently obtained by G. Auchmuty. In both problems the admissible set is the one comprising characteristic functions whose supports have a fixed measure. We prove the maximization to be solvable, whilst the minimization will turn out not to be solvable in general. We will also discuss the case of radial domains. In the second part of the paper, we study approximation and stability results regarding rearrangement optimization problems. First, we show that if a sequence of the generators of rearrangement classes converges, then the corresponding sequence of the optimal solutions will also converge. Second, a stability result regarding the Hausdorff distance between the weak closures of two rearrangement classes is presented.

Key Words: Trace inequality, Boundary value problem, Maximization, Minimization, Approximation, Stability, Rearrangement theory

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1 Introduction

In [2], Auchmuty describes some sharp trace inequalities, amongst which one finds

$$\int_{\partial D} \rho |u|\, d\mathcal{H}^{N-1} \leq k_\rho \int_D |u|\, dx + \|\nabla u_\rho\|_2 \, \|\nabla u\|_2, \quad \forall u \in H^1(D), \tag{1.1}$$

under the following regime:

(1) D is a finite union of smooth surfaces in $\mathbb{R}^N$ ($N \geq 2$).[1]

(2) The *weight function* ρ belongs to $L^s(\partial D)$, in which

$$\begin{cases} \infty > s \geq s_c = 2(1 - \frac{1}{N}) & \text{if } N \geq 3 \\ \infty > s > 1 & \text{if } N = 2. \end{cases}$$

(3) $k_\rho = \frac{1}{|D|} \int_{\partial D} \rho \, d\mathcal{H}^{N-1}$.

(4) The *trace weight function* u_ρ is the solution of the following boundary value problem:

$$\begin{cases} \Delta u = \dfrac{1}{|D|} \displaystyle\int_{\partial D} \rho \, d\mathcal{H}^{N-1} & \text{in } D, \\ \dfrac{\partial u}{\partial \nu} = \rho & \text{on } \partial D, \\ \displaystyle\int_D u \, dx = 0, \end{cases} \tag{1.2}$$

where $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative to the boundary.

Notation 1.1. Henceforth, $d\mathcal{H}^{N-1}$ denotes the $N-1$-dimensional Hausdorff measure on ∂D, and D, ρ, s, u_ρ is described as above.

A number of trace estimates can be derived by inserting different weight functions in (1.1). In particular, the following cases have been mentioned in [2]:

[1] The actual condition in [2] is slightly less restrictive.

- By choosing $\rho = 1$, one can derive estimates for the norms of the trace operator $\Gamma : W^{1,r}(D) \to L^p(\partial D)$.
- If ρ is taken to be the characteristic function of proper subsets of the boundary, then inequality (1.1) will provide restricted trace estimates.

In the first part of the paper we will develop the discussion of the inequality (1.1) further for the case where ρ ranges over a particular class $\mathcal{S}$ of characteristic functions

$$\mathcal{S} = \left\{ \rho \in L^s(\partial D) \mid \rho(\rho - 1) = 0, \int_{\partial D} \rho \, d\mathcal{H}^{N-1} = \beta \right\}, \tag{1.3}$$

with further assumptions of $\beta > 0$ and $\mathcal{H}^{N-1}(\partial D) > 0$. We explore the possibility of improving the inequality (1.1) in the following sense: *Is it possible to find a weight function $\rho \in \mathcal{S}$ for which $\|\nabla u_\rho\|_2$ is minimal?* We shall prove that when D is a ball, the answer is negative.

We will also address the question of whether there is a $\rho \in \mathcal{S}$ for which $\|\nabla u_\rho\|_2$ is maximal. In contrast to the previous case, the answer to this question is affirmative in any type of domain. However, in case D is a ball, we will see that it is possible to find an optimal solution which is spherically symmetric.

Let us set $\alpha(\rho) = \|\nabla u_\rho\|_2^2$. We are interested in the following optimization problems:

$$\sup_{\rho \in \mathcal{S}} \alpha(\rho), \tag{1.4}$$

and

$$\inf_{\rho \in \mathcal{S}} \alpha(\rho). \tag{1.5}$$

If we consider a specific subset $\hat{E}$ of ∂D such that $\mathcal{H}^{N-1}(\hat{E}) = \beta$, then $\mathcal{S}$ would be the rearrangement class[2] generated by the characteristic function

$$\chi_{\hat{E}}(x) = \begin{cases} 1, & x \in \hat{E} \\ 0, & x \notin \hat{E}. \end{cases}$$

It is common to write $\mathcal{S} = \mathcal{R}(\chi_{\hat{E}})$. For instance, the minimization problem (1.5) can be written as

$$\inf_{\rho \in \mathcal{R}(\chi_{\hat{E}})} \alpha(\rho). \tag{1.6}$$

In the second part of the paper we look at a more general rearrangement problem than (1.6) and describe an approximation scheme. Let us elaborate this as follows. Recently, by using the well established rearrangement theory attributed

[2]See Definition 2.4 on page 10.

to G. R. Burton, a significant amount of research (e. g. [7, 10] and references therein) has been focused on the following type of rearrangement optimization problem (ROP):

$$\inf_{f\in\mathcal{R}(f_0)} \Phi(f), \tag{1.7}$$

in which Φ is a nonlinear functional related to a partial differential equation, f_0 is a non-negative function in an appropriate L^p space, and $\mathcal{R}(f_0)$ is the rearrangement class generated by f_0.

Let us present two examples of (1.7) from [12] and [16]. Henceforth, D will denote a smooth bounded domain in $\mathbb{R}^N$.

Example 1.1. Consider the boundary value problem

$$\begin{cases} -\Delta_p u = f & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1.8}$$

where:

- Δ_p is the classical p-Laplace operator, i. e. $\Delta_p u = \nabla\cdot(|\nabla u|^{p-2}\nabla u)$, with $1 < p < \infty$.
- $f \in L^{p'}(D)$ in which p' is the conjugate exponent of p, i. e. $\frac{1}{p} + \frac{1}{p'} = 1$.

Denoting the unique solution of (1.8) by $u_f \in W_0^{1,p}(D)$, we are interested in the following rearrangement optimization problem:

$$\inf_{f\in\mathcal{R}(f_0)} \Phi_1(f) \equiv \int_D f u_f\, dx, \tag{1.9}$$

for some non-negative generator $f_0 \in L^{p'}(D)$.

Example 1.2. Assume that $g_0 : D \to \mathbb{R}$ satisfies $g_0(x) \in [0, 1]$ a. e. in D, and let $g \in \mathcal{R}(g_0)$. Consider the boundary value problem

$$\begin{cases} -\Delta u + gu = f & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1.10}$$

where f is a non-negative function in $L^2(D)$. Denoting the unique solution of (1.10) by $u_g \in H_0^1(D)$, we are interested in the following rearrangement optimization problem:

$$\inf_{g\in\mathcal{R}(g_0)} \Phi_2(g) \equiv \int_D f u_g\, dx. \tag{1.11}$$

The existence and uniqueness of solutions of (1.9) and (1.11) in their designated rearrangement classes have already been established in [12] and [16], respectively.

Consider the problem (1.7) and for a generator f_0, let $\hat{f}_0$ be the unique minimizer of Φ. In this paper we will answer the following question:

Question 1.1. If a sequence of generators (f_n) converges to f in an appropriate L^p space, does the sequence of minimizers $(\hat{f}_n)$ also converge to $\hat{f}$ in the same space?

Aside from its theoretical implications, this question has practical relevance too. In practice, it is not always possible to find an explicit exact solution of the rearrangement optimization problem (1.7), in which case, one would apply numerical simulations. If the generator of the rearrangement class is a simple function, it will simplify the computations accordingly. As every measurable function can be approximated by simple functions, an answer to Question 1.1 will be significant in approximating the real solution.

Another reason adding to the importance of the above question is that during the process of proving the existence of minimizers for (1.7), the zero level set of the generator, i. e. $\{x \in D \mid f_0(x) = 0\}$, is a tricky one to deal with. Thus, it would be helpful if we could approximate the generator by positive functions.

The weak closure of a rearrangement class is of great importance in rearrangement theory.[3] Motivated by Question 1.1, it is interesting to address the following stability question as well:

Question 1.2. If f_n converges to f in an appropriate L^p space, does the Hausdorff distance[4] between the weak closures $\overline{\mathcal{R}(f_n)}^w$ and $\overline{\mathcal{R}(f)}^w$ also tend to zero?

Structure of the paper: The paper is organized as follows. In Section 2 we collect some well-known results mostly from the theory of rearrangements of functions. Section 3 contains the main results where answers to the two aforementioned questions will be presented.

2 Preliminaries

Let us introduce the space

$$H^1_m(D) = \left\{ u \in H^1(D) \mid \int_D u \, dx = 0 \right\}.$$

[3]See item (ii) of Lemma 2.5 on page 10.

[4]See Definition 2.5 on page 11.

which is a closed subspace of $H^1(D)$. As a result, it is a Hilbert space. By using the Poincaré inequality specialized to functions with zero mean, we infer that $H^1_m(D)$ remains a Hilbert space with respect to the norm $\|u\| := \|\nabla u\|_2$.

Definition 2.1. We say that $u \in H^1_m(D)$ is a solution of (1.2) provided that the following integral equation holds:

$$\int_D \nabla u \cdot \nabla v \, dx - \int_{\partial D} \rho v \, d\mathcal{H}^{N-1} = 0, \quad \forall v \in H^1_m(D). \tag{2.1}$$

The energy functional $\mathcal{E} : H^1_m(D) \to \mathbb{R}$ associated with (1.2) is:

$$\mathcal{E}(u) = \frac{1}{2}\int_D |\nabla u|^2 \, dx - \int_{\partial D} \rho u \, d\mathcal{H}^{N-1}.$$

The trace embedding $H^1_m(D) \to L^{s'}(\partial D)$ guarantees that $\forall u \in H^1_m(D) : \mathcal{E}(u) \in \mathbb{R}$.

The following result is a basic one:

Theorem 2.1. *The following statements hold:*

(i) The minimization problem

$$\inf_{u \in H^1_m(D)} \mathcal{E}(u) \tag{2.2}$$

has a unique solution.

(ii) The function $u \in H^1_m(D)$ is a solution of (1.2) if and only if it is a solution of (2.2). In particular, (1.2) has a unique solution.

Proof. (i) By trace embedding [1] we have $H^1(D) \to L^{s'}(\partial D)$, where s' is the conjugate exponent of s. Thus:

$$\begin{aligned} \mathcal{E}(u) &= \frac{1}{2}\int_D |\nabla u|^2 \, dx - \int_{\partial D} \rho u \, d\mathcal{H}^{N-1} \\ \text{(Hölder's inequality)} \quad &\geq \frac{1}{2}\int_D |\nabla u|^2 \, dx - \|\rho\|_{s,\partial D} \|u\|_{s',\partial D} \\ \text{(trace embedding and Poincaré)} \quad &\geq \frac{1}{2} C_1 \|u\|^2_{H^1(D)} - C_2 \|\rho\|_{s,\partial D} \|u\|_{H^1(D)}. \end{aligned}$$

This shows that $\mathcal{E}(\cdot)$ is coercive. As a result, if (u_n) is a minimizing sequence of (2.2), we infer the existence of a subsequence—still denoted (u_n)—and a function $u \in H^1(D)$ such that $u_n \rightharpoonup u$ in $H^1(D)$ and $u_n \rightharpoonup u$ in $L^{s'}(\partial D)$. It should be clear that $u \in H^1_m(D)$ and $\mathcal{E}(u) \leq \liminf_{n\to\infty} \mathcal{E}(u_n) = \inf_{v \in H^1_m(D)} \mathcal{E}(v)$. As $\mathcal{E}(\cdot)$ is strictly convex, then u must be the unique solution of (2.2).

(ii) Let u be the solution of (2.2). Then $\langle \mathcal{E}'(u), v\rangle = 0$, for every $v \in H^1_m(D)$. Hence, by definition, u is a solution of (1.2).

Conversely, let u be a solution of (1.2). Then, for all $v \in H^1_m(D)$ we have:

$$\begin{aligned} 0 &= \langle \mathcal{E}'(u), v - u\rangle \\ &= \lim_{t\to 0^+} \frac{\mathcal{E}(u + t(v-u)) - \mathcal{E}(u)}{t} \\ \text{(as } \mathcal{E} \text{ is convex)} &\le \mathcal{E}(v) - \mathcal{E}(u). \end{aligned} \tag{2.3}$$

From (2.3), we deduce that $\mathcal{E}(u) \le \mathcal{E}(v)$, for every $v \in H^1_m(D)$. So, u is a solution of (2.2). □

The admissible set $\mathcal{S}$, as it is, is not particularly convenient to deal with in the context of optimization. The common trick is to enlarge $\mathcal{S}$ to a superset $\mathcal{A}$:

$$\mathcal{S} \subseteq \mathcal{A} := \left\{ \rho \in L^s(\partial D) \mid 0 \le \rho \le 1, \int_{\partial D} \rho \, d\mathcal{H}^{N-1} = \beta \right\}. \tag{2.4}$$

The following connections between $\mathcal{S}$ and $\mathcal{A}$ are widely known, see Lemmata 2.2 and 2.3 in [5] for details:

(i) $\mathcal{A} = \overline{\mathcal{S}}^w$, i. e. the weak closure of $\mathcal{S}$ in $L^s(\partial D)$.

(ii) $\mathcal{A}$ is weakly compact and convex.

(iii) $\mathcal{S} = \mathrm{ext}(\mathcal{A})$, the set of extreme points of $\mathcal{A}$.

(iv) $\mathcal{A} = \overline{co}\, \mathcal{S}$, the closed convex hull of $\mathcal{S}$.

The next result is crucial.

Theorem 2.2. *Let $\Phi : L^s(\partial D) \to \mathbb{R}$ be strictly convex and weakly sequentially continuous. Then the maximization problem*

$$\sup_{f\in\mathcal{S}} \Phi(f) \tag{2.5}$$

is solvable, i. e. there exists $\hat{f} \in \mathcal{S}$ such that $\Phi(\hat{f}) = \sup_{f\in\mathcal{S}} \Phi(f)$.

Proof. We begin by relaxing the maximization problem (2.5) and then we consider the problem:

$$\sup_{f\in\mathcal{A}} \Phi(f) \tag{2.6}$$

Since Φ is weakly continuous, and $\mathcal{A}$ is weakly compact, (2.6) is solvable. Let $\overline{f} \in \mathcal{A}$ be a solution. As Φ is convex and continuous, it is subdifferentiable at $\overline{f}$, see Proposition 4.6 in [6]. Hence, $\partial\Phi(\overline{f}) \neq \emptyset$, in which:

$$\partial\Phi(\overline{f}) = \left\{ g \in L^{s'}(\partial D) \mid \Phi(f) \geq \Phi(\overline{f}) + \int_{\partial D} g(f - \overline{f})\, d\mathcal{H}^{N-1},\ \forall f \in L^s(\partial D) \right\}$$

Let $\overline{g} \in \partial\Phi(\overline{f})$. It is known (e. g. Theorem 4 in [4]) that the linear functional $L(h) := \int_{\partial D} \overline{g} h\, d\mathcal{H}^{N-1}$ has a maximizer $\overline{h}$ relative to $\mathcal{S}$. Since L is weakly continuous, it follows that $\overline{h}$ maximizes L relative to $\mathcal{A}$ as well. Whence, in particular, $L(\overline{f}) \leq L(\overline{h})$. By subdifferentiability we have:

$$\begin{aligned} \Phi(\overline{h}) \geq \Phi(\overline{f}) + \int_{\partial D} \overline{g}(\overline{h} - \overline{f})\, d\mathcal{H}^{N-1} &= \Phi(\overline{f}) + L(\overline{h}) - L(\overline{f}) \\ &\geq \Phi(\overline{f}) \geq \Phi(\overline{h}). \end{aligned}$$

Thus, $\Phi(\overline{f}) = \Phi(\overline{h})$ and $\overline{h} \in \mathcal{S}$ is a solution of (2.6), as desired. □

Since the solution of (1.2) is unique we can define the operator $K : L^s(\partial D) \to H^1_m(D)$ by $K(\rho) = u_\rho$.

Lemma 2.3. *The following statements are true.*

(i) K is linear.

(ii) K is symmetric in the sense that:

$$\int_{\partial D} \rho_1 K\rho_2\, d\mathcal{H}^{N-1} = \int_{\partial D} \rho_2 K\rho_1\, d\mathcal{H}^{N-1}, \quad \forall\, \rho_1, \rho_2 \in L^s(\partial D). \tag{2.7}$$

Proof. Assertion (i) follows from the linear nature of the boundary value problem (1.2). For assertion (ii), we choose $u = K\rho_1 = u_{\rho_1}$ and $v = K\rho_2 = u_{\rho_2}$ in (2.1) to obtain:

$$\int_D \nabla(K\rho_1) \cdot \nabla(K\rho_2)\, dx - \int_{\partial D} \rho_1 K\rho_2\, d\mathcal{H}^{N-1} = 0.$$

from which (2.7) follows. □

Lemma 2.4. *Remember that we defined $\alpha(\rho) = \|\nabla u_\rho\|_2^2$:*

(i) α is weakly sequentially continuous in $L^s(\partial D)$.

(ii) α is strictly convex.

(iii) α is Gâteaux differentiable, moreover, $\alpha'(\rho)$ can be identified with $2K\rho$.

Proof.

(i) Let $\{\rho_n\} \subseteq L^s(\partial D)$ such that $\rho_n \rightharpoonup \rho$ in $L^s(\partial D)$. By using (2.1), Hölder's inequality, and trace embedding theory, we infer that:

$$\begin{aligned}\int_D |\nabla K\rho_n|^2\,dx &= \int_{\partial D} \rho_n K\rho_n\,d\mathcal{H}^{N-1}\\ &\leq \|\rho_n\|_{s,\partial D}\,\|K\rho_n\|_{s',\partial D}\\ &\leq C\,\|\rho_n\|_{s,\partial D}\,\|K\rho_n\|_{H^1(D)}\,.\end{aligned}$$

Since $\rho_n \rightharpoonup \rho$ in $L^s(\partial D)$ and $K\rho_n \in H^1_m(D)$, it follows from Poincaré inequality that $\{K\rho_n\}$ is bounded. Therefore, we infer the existence of a subsequence, still denoted $\{K\rho_n\}$, such that $K\rho_n \rightharpoonup w$ in $H^1(D)$ and $K\rho_n \to w$ in $L^{s'}(\partial D)$. So, by invoking (2.1), we deduce that

$$\int_D \nabla w \cdot \nabla v\,dx - \int_{\partial D} \rho v\,d\mathcal{H}^{N-1} = 0, \quad \forall\, v \in H^1_m(D),$$

which means that w is a critical point of the energy functional $\mathcal{E}(\cdot)$. By strict convexity of $\mathcal{E}$, we must have $w = K\rho$. By applying (2.1) again, we have

$$\alpha(\rho_n) = \int_D |\nabla K\rho_n|^2\,dx = \int_{\partial D} \rho_n K\rho_n\,d\mathcal{H}^{N-1}$$

which implies that:

$$\lim_{n\to\infty} \alpha(\rho_n) = \int_{\partial D} \rho w\,d\mathcal{H}^{N-1} = \int_{\partial D} \rho K\rho\,d\mathcal{H}^{N-1} = \alpha(\rho).$$

(ii) By using Lemma 2.3 (i), the strict convexity of α follows from its equivalent form $\alpha(\rho) = \|\nabla K\rho\|_2^2$.

(iii) Let us fix $\hat{\rho}$ and ρ in $L^s(\partial D)$. By applying (2.1) and Lemma 2.3, we have:

$$\begin{aligned}\alpha(\hat{\rho} + t\rho) &= \int_{\partial D} (\hat{\rho} + t\rho)K(\hat{\rho} + t\rho)\,d\mathcal{H}^{N-1}\\ &= \int_{\partial D} \hat{\rho}K\hat{\rho}\,d\mathcal{H}^{N-1} + 2t\int_{\partial D} \rho K\hat{\rho}\,d\mathcal{H}^{N-1} + t^2\int_{\partial D} \rho K\rho\,d\mathcal{H}^{N-1}\\ &= \alpha(\hat{\rho}) + 2t\int_{\partial D} \rho K\hat{\rho}\,d\mathcal{H}^{N-1} + t^2\alpha(\rho).\end{aligned}$$

This implies that:

$$\lim_{t\to 0^+} \frac{\alpha(\hat{\rho} + t\rho) - \alpha(\hat{\rho})}{t} = 2\int_{\partial D} \rho K\hat{\rho}\,d\mathcal{H}^{N-1},$$

which is the desired result. □

Henceforth, for a measurable set E, $|E|$ denotes the N-dimensional Lebesgue measure of E. Moreover, for a Lebesgue measurable function $f : D \to [0,\infty)$, its distribution function $\lambda_f : \mathbb{R} \to \mathbb{R}$ is defined as:

$$\lambda_f(\alpha) = |\{x \in D : f(x) \geq \alpha\}|.$$

Definition 2.2. Let $f, f_0 : D \to [0,\infty)$ be Lebesgue measurable. We say that f is a *rearrangement* of f_0 if and only if $\forall \alpha \in [0,\infty) : \lambda_{f_0}(\alpha) = \lambda_f(\alpha)$.

Definition 2.3. For a Lebesgue measurable $f : D \to [0,\infty)$, the essentially unique decreasing rearrangement f^Δ is defined on $(0,|D|)$ by $f^\Delta(s) = \sup\left\{\alpha : \lambda_f(\alpha) \geq s\right\}$. The essentially unique increasing rearrangement f_Δ of f is defined by $f_\Delta(s) = f^\Delta(|D| - s)$.

Definition 2.4. The rearrangement class $\mathcal{R}(f)$ generated by f is defined as:

$$\mathcal{R}(f) := \{g : D \to [0,\infty) \mid g \text{ is a rearrangement of } f\}.$$

The following are two basic results regarding rearrangement classes and essentially unique decreasing rearrangements.

Lemma 2.5. *Let $1 \leq p < \infty$, $f \in L^p(D)$, and $\mathcal{R} \equiv \mathcal{R}(f)$ be the rearrangement class generated by f. Then:*

(i) $\mathcal{R} \subseteq L^p(D)$, and $\|f\|_p = \|g\|_p$ for all $g \in \mathcal{R}$. Here $\|\cdot\|_p$ denotes the usual L^p-norm.

(ii) The weak closure $\overline{\mathcal{R}}^w$ of $\mathcal{R}$ in $L^p(D)$ is convex and weakly (sequentially) compact in $L^p(D)$. Moreover, $\overline{\mathcal{R}}^w$ is the closed convex hull of $\mathcal{R}$ in $L^p(D)$, which we write as $\overline{\mathcal{R}}^w = \overline{co}(\mathcal{R})$.

(iii) The relative weak and strong topologies on $\mathcal{R}$ coincide.

Proof. See Lemma 2.1, Lemma 2.2 and Lemma 2.6 in [5]. □

Lemma 2.6. *Let p, f and $\mathcal{R}$ be as in Lemma 2.5. Then:*

(i) There is a measure preserving map $\rho : D \to (0,|D|)$ such that $f = f^\Delta \circ \rho$.

(ii) $\left\|g^\Delta - h^\Delta\right\|_p \leq \|g - h\|_p$ for all g and h in $L^p(D)$.

(iii) The weak closure of $\mathcal{R}$ has the following characterization:

$$\overline{\mathcal{R}}^w = \left\{g \in L^1(D) \mid \int_D g\,dx = \int_D f\,dx \right.$$
$$\left. \text{and } \forall s \in (0,|D|) : \int_0^s g^\Delta dt \leq \int_0^s f^\Delta dt\right\}.$$

Proof. For (i), see Lemma 2.4 in [5] or Proposition 3 in [18]. For (ii), see Lemma 2.7 in [5] or Corollary 1 in [8]. For (iii), see Lemma 2.3 in [5]. □

The concept of Hausdorff distance will also be refered to later:

Definition 2.5. Let (X, d) be a metric space. Suppose that L and K are two non-empty subsets of X. Then, the Hausdorff distance between L and K is defined by

$$d_H(L, K) = \max\left\{\sup_{x\in K}\left(\inf_{y\in L} d(x, y)\right), \sup_{y\in L}\left(\inf_{x\in K} d(x, y)\right)\right\}.$$

Finally, let us recall the Radon-Riesz Theorem.

Theorem 2.7. *Let* $1 < p < \infty$, $f \in L^p(D)$ *and* $\{f_n\} \subseteq L^p(D)$. *If* $f_n \rightharpoonup f$ *in* $L^p(D)$ *and* $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$, *then* $f_n \to f$ *in* $L^p(D)$.

Proof. See section 37 in [17]. □

3 Maximization and minimization problems

3.1 Maximization problem (1.4)

In this subsection, we focus on the maximization problem (1.4) and the first main result is the following:

Theorem 3.1. *The maximization problem (1.4) is solvable, i. e. there exists a weight function* $\hat{\rho} \in \mathcal{S}$ *for which:*

$$\alpha(\hat{\rho}) = \sup_{\rho\in\mathcal{S}} \alpha(\rho).$$

Proof. Since $\alpha : L^s(\partial D) \to \mathbb{R}$ is strictly convex and weakly sequentially continuous, the assertion follows immediately from Theorem 2.2. □

Remark 3.1. As α is strictly convex, one can argue as in Theorem 7 of [4] to show that if ρ is any solution of (1.4), then there exists a non-decreasing function $\phi : \mathbb{R} \to \{0, 1\}$ such that $\rho = \phi \circ u_\rho$ $\mathcal{H}^{N-1}$ a. e. on ∂D.

Before stating the second main result of this section, we need to recall the *spherical symmetrization*. For detailed treatment of this topic, we refer the reader to [3, 11, 14, 19].[5] Given a measurable set $K \subseteq \mathbb{R}^N$, we fix a direction $\vec{e}$ with $|\vec{e}| = 1$. Then, the spherical symmetrization of K with respect to direction $\vec{e}$,

[5]Note that, in Section 3 of [3] the author presents a variety of properties of a general class of rearrangements. Spherical symmetrization belongs to this class.

denoted by K^*, is characterized by the following property: for every $r \in (0,\infty)$, the set $K^* \cap \partial B(0,r)$ is a spherical cap centered at $r\vec{e}$ satisfying:

$$\mathcal{H}^{N-1}\left(K^* \cap \partial B(0,r)\right) = \mathcal{H}^{N-1}\left(K \cap \partial B(0,r)\right), \quad \forall\, 0 < r < \infty,$$

where $B(0,r)$ is an open ball centered at the origin with radius r. For a non-negative measurable function u, the spherical symmetrization u^* is constructed such that:

$$\{u^* \geq t\} = \{u \geq t\}^*, \quad \forall\, t \geq 0.$$

Theorem 3.2. *Let D be a ball centered at origin with radius a. Then, for any direction $\vec{e}$, the maximization problem (1.4) has a solution $\hat{\rho} \in \mathcal{S}$ which is spherically symmetrized with respect to it.*

Before proving the theorem, we need the following known results about spherical symmetrization.

Lemma 3.3. *Let D be as in Theorem 3.2, and u be a non-negative function in $H^1(D)$. Then, we have:*

(i) $u^* \in H^1(D)$.

(ii) *For any non-negative $v \in L^{s'}(\partial D)$, we have* $\int_{\partial D} uv\, d\mathcal{H}^{N-1} \leq \int_{\partial D} u^* v^*\, d\mathcal{H}^{N-1}$.

(iii) $\int_D |\nabla u^*|^2\, dx \leq \int_D |\nabla u|^2\, dx$.

(iv) *For any $v \in L^s(\partial D)$, we have* $\int_{\partial D} |v^* - u^*|^s\, d\mathcal{H}^{N-1} \leq \int_{\partial D} |v - u|^s\, d\mathcal{H}^{N-1}$.

We also need the following regularity result of the solution of the Neumann boundary value problem (1.2).

Lemma 3.4. *If $\rho \in C^\infty(\partial D)$, then $K\rho \in C^\infty(\bar{D})$.*

Proof. Since $\rho \in C^\infty(\partial D) \subseteq H^k(\partial D)$ for all $k \geq 0$, by Proposition 7.1 in [20], we have $K\rho \in H^{k+2}(D)$ for all $k \geq 0$. Then, by applying Sobolev embedding theorem (Theorem 4.12 in [1]), we infer that $K\rho \in C^\infty(\bar{D})$. □

Proof of Theorem 3.2: Let $\hat{\rho}$ be any maximizer of (1.4) in $\mathcal{S}$, whose existence is guaranteed by Theorem 3.1. Then, there exists a sequence $\{\rho_n\} \subseteq C^\infty(\partial D)$ such that $\rho_n \to \hat{\rho}$ in $L^s(\partial D)$ (by mollifiers). Observing that, from Theorem 2.1 and (2.1), we have

$$\begin{aligned}\alpha(\rho) &= 2\int_{\partial D} \rho K\rho\, d\mathcal{H}^{N-1} - \int_D |\nabla K\rho|^2\, dx \\ &= \sup_{v \in H^1_m(D)} \left\{ 2\int_{\partial D} \rho v\, d\mathcal{H}^{N-1} - \int_D |\nabla v|^2\, dx \right\}.\end{aligned} \tag{3.1}$$

Then, let us fix any direction $\vec{e}$ and we will perform the spherical symmetrization with respect to this direction. From (3.1) and Lemma 3.4, we infer that

$$
\begin{aligned}
\alpha(\rho_n) &= 2\int_{\partial D} \rho_n K\rho_n \, d\mathcal{H}^{N-1} - \int_D |\nabla K\rho_n|^2 \, dx \\
&= 2\int_{\partial D} \rho_n (K\rho_n + \|K\rho_n\|_{\infty,\bar{D}}) \, d\mathcal{H}^{N-1} \\
&\quad - \int_D |\nabla (K\rho_n + \|K\rho_n\|_{\infty,\bar{D}})|^2 \, dx - 2\,\|K\rho_n\|_{\infty,\bar{D}} \int_{\partial D} \rho_n \, d\mathcal{H}^{N-1} \\
&\le 2\int_{\partial D} \rho_n^* ((K\rho_n)^* + \|K\rho_n\|_{\infty,\bar{D}}) \, d\mathcal{H}^{N-1} \\
&\quad - \int_D |\nabla ((K\rho_n)^* + \|K\rho_n\|_{\infty,\bar{D}})|^2 \, dx - 2\,\|K\rho_n\|_{\infty,\bar{D}} \int_{\partial D} \rho_n \, d\mathcal{H}^{N-1} \\
&= 2\int_{\partial D} \rho_n^* (K\rho_n)^* \, d\mathcal{H}^{N-1} - \int_D |\nabla (K\rho_n)^*|^2 \, dx \\
&\le 2\int_{\partial D} \rho_n^* K\rho_n^* \, d\mathcal{H}^{N-1} - \int_D |\nabla K\rho_n^*|^2 \, dx = \alpha(\rho_n^*),
\end{aligned}
\tag{3.2}
$$

where we have used Lemma 3.3 in the first inequality. On the other hand, by Lemma 3.3 (iv), we have $\rho_n^* \to \hat{\rho}^*$ in $L^s(\partial D)$. So, due to the continuity of α, it follows from (3.2) that $\alpha(\hat{\rho}) \le \alpha(\hat{\rho}^*)$. Recalling that $\hat{\rho}$ is a maximizer of (1.4), $\hat{\rho}^*$ must be a maximizer as well. As the direction $\vec{e}$ was chosen arbitrarily, the proof of the theorem is complete. □

3.2 Minimization problem (1.5)

The following is the first main result of this subsection:

Theorem 3.5. *The minimization problem (1.5) has a unique solution $\tilde{\rho} \in \mathcal{A}$.*

Proof. As in the proof of Theorem 2.2, we have $\mathcal{A} = \overline{\mathcal{R}(\chi_E)}^w$ with $\mathcal{H}^{N-1}(E) = \beta$, where the weak closure is taken in $L^s(\partial D)$. Then, by Lemma 2.5 (ii), $\mathcal{A}$ is convex and weakly sequentially compact. As by Lemma 2.4 the function α is strictly convex and weakly sequentially continuous, there must exist a unique solution $\tilde{\rho} \in \mathcal{A}$ which solves (1.5). □

As opposed to the case of maximization, in general the solution of (1.5) is not in $\mathcal{S}$. In particular, we have the following:

Theorem 3.6. *Let D be a ball centered at origin with radius a. Then, the minimization problem (1.5) has a unique solution $\tilde{\rho} \in \mathcal{A} \setminus \mathcal{S}$. In fact, this unique solution is the constant function:*

$$\forall x \in \partial D : \; \tilde{\rho}(x) = \frac{\beta}{\mathcal{H}^{N-1}(\partial D)}.$$

For the proof of Theorem 3.6, we need the following result:

Lemma 3.7. *Let D be as in Theorem 3.6, and assume that $\rho \in L^s(\partial D)$. For any rotation map R about the origin, we have $K(\rho \circ R) = (K\rho) \circ R$ almost everywhere in B.*

Proof. By Theorem 2.1, equation (2.1), and also Theorem 3.22 in [1], it suffices to show that

$$\int_D \nabla((K\rho) \circ R)(x) \cdot \nabla v(x)\, dx - \int_{\partial D} (\rho \circ R)(x)\, v(x)\, d\mathcal{H}^{N-1}(x) = 0, \qquad \forall\, v \in H^1_m(D) \cap C^\infty(D).$$

Let us fix an arbitrary $v \in H^1_m(D)$. Since $\nabla((K\rho) \circ R)(x) = R^{-1}\nabla(K\rho)(Rx)$ and $R^{-1} = R^t$, the transpose of R, we deduce that:

$$\begin{aligned}
\mathfrak{I} &\equiv \int_D \nabla((K\rho) \circ R)(x) \cdot \nabla v(x)\, dx - \int_{\partial D} (\rho \circ R)(x)\, v(x)\, d\mathcal{H}^{N-1}(x) \\
&= \int_D R^{-1}\nabla(K\rho)(Rx) \cdot \nabla v(x)\, dx - \int_{\partial D} (\rho \circ R)(x)\, v(x)\, d\mathcal{H}^{N-1}(x) \\
&= \int_D \nabla(K\rho)(Rx) \cdot R\nabla v(x)\, dx - \int_{\partial D} (\rho \circ R)(x)\, v(x)\, d\mathcal{H}^{N-1}(x).
\end{aligned}$$

Similarly, we also have $R\nabla v(x) = \nabla(v \circ R^{-1})(Rx)$. By a change of variables $y = Rx$, the equation above leads to

$$\begin{aligned}
\mathfrak{I} &= \int_D \nabla(K\rho)(Rx) \cdot \nabla(v \circ R^{-1})(Rx)\, dx - \int_{\partial D} (\rho \circ R)(x)\, v(x)\, d\mathcal{H}^{N-1}(x) \\
&= \int_D \nabla(K\rho)(y) \cdot \nabla(v \circ R^{-1})(y) dy - \int_{\partial D} \rho(y)(v \circ R^{-1})(y)\, d\mathcal{H}^{N-1}(y).
\end{aligned}$$

Now, because $\nabla(v \circ R^{-1})(x) = R\nabla v(R^{-1}x)$ and $|\nabla(v \circ R^{-1})(x)| = |\nabla v(R^{-1}x)|$, we infer that $v \circ R^{-1} \in H^1_m(D) \cap C^\infty(D)$. By (2.1), we must have $\mathfrak{I} = 0$ as desired. □

Proof of Theorem 3.6: Fix any rotation R and let $\rho \in \mathcal{S}$. Then, by (2.1) and Lemma 3.7 we have:

$$\begin{aligned}
\alpha(\rho \circ R) = \int_{\partial D} (\rho \circ R) K(\rho \circ R)\, d\mathcal{H}^{N-1} &= \int_{\partial D} (\rho \circ R)((K\rho) \circ R)\, d\mathcal{H}^{N-1} \\
&= \int_{\partial D} \rho K\rho\, d\mathcal{H}^{N-1} = \alpha(\rho).
\end{aligned} \tag{3.3}$$

Let $\tilde{\rho} \in \mathcal{A}$ be the unique solution of (1.5) (see Theorem 3.5). Then, we fix an arbitrary axis ℓ (through origin), and consider the rotation map R_θ with which $\tilde{\rho}$ can be rotated to $\tilde{\rho} \circ R_\theta$ by the angle $-\theta$ with respect to the axis ℓ.

Now consider the constant function $\mathfrak{C} : \partial D \to \mathbb{R}$ defined by:

$$\forall x \in \partial D : \ \mathfrak{C}(x) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\rho} \circ R_\theta \, d\theta$$

It should be clear that $\mathfrak{C} \in \mathcal{A}$. Now, as α is strictly convex, by (3.3) and Jensen's inequality (see e. g. [13]) we deduce that:

$$\alpha(\mathfrak{C}) \leq \frac{1}{2\pi} \int_0^{2\pi} \alpha(\tilde{\rho} \circ R_\theta) \, d\theta = \alpha(\tilde{\rho}). \tag{3.4}$$

Since (3.4) is true for rotations with respect to any axis through origin, then $\tilde{\rho}$ must be radially symmetric. As $\tilde{\rho} \in \mathcal{A}$, we must have $\tilde{\rho} = \beta / \mathcal{H}^{N-1}(\partial D)$ which is obviously not in $\mathcal{S}$. □

Remark 3.2. Theorem 3.6 implies that (1.5) is not solvable in radial domains. However, it might still be possible for (1.5) to be solvable in non-radial domains. Let us elaborate this matter. Let $\tilde{\rho} \in \mathcal{A}$ be the unique solution of (1.5). Then, $\tilde{\rho}$ satisfies the optimality condition

$$0 \in \partial\alpha(\tilde{\rho}) + \mathcal{N}_{\mathcal{A}}(\tilde{\rho}), \tag{3.5}$$

where $\mathcal{N}_{\mathcal{A}}(\tilde{\rho})$ denotes the normal cone to $\mathcal{A}$ at $\tilde{\rho}$. Since $\mathcal{A}$ is convex, $\mathcal{N}_{\mathcal{A}}(\tilde{\rho}) = \partial\xi_{\mathcal{A}}(\tilde{\rho})$, where $\xi_{\mathcal{A}}(\tilde{\rho})$ is the indicator function

$$\xi_{\mathcal{A}}(\rho) = \begin{cases} 0 & \text{if } \rho \in \mathcal{A} \\ \infty & \text{if } \rho \notin \mathcal{A} \end{cases}$$

supported on $\mathcal{A}$. So,

$$\partial\xi_{\mathcal{A}}(\tilde{\rho}) = \left\{ g \in L^{s'}(\partial D) \mid \xi_{\mathcal{A}}(\rho) \geq \xi_{\mathcal{A}}(\tilde{\rho}) + \int_{\partial D} g(\rho - \tilde{\rho}) \, d\mathcal{H}^{N-1}, \ \forall \rho \in L^s(\partial D) \right\}.$$

Recall that $\partial\alpha(\tilde{\rho}) = \{2\tilde{u}\}$, where $\tilde{u} = u_{\tilde{\rho}}$. So, (3.5) implies that

$$\exists g \in \partial\xi_{\mathcal{A}}(\tilde{\rho}) : \quad 2\tilde{u} + g = 0, \quad \mathcal{H}^{N-1} \text{ a. e. on } \partial D, \tag{3.6}$$

Clearly, for any $\rho \in \mathcal{A}$, $\int_{\partial D} g(\tilde{\rho} - \rho) \, d\mathcal{H}^{N-1} \geq 0$. Whence, (3.6) yields

$$\int_{\partial D} \rho\tilde{u} \, d\mathcal{H}^{N-1} \geq \int_{\partial D} \tilde{\rho}\tilde{u} \, d\mathcal{H}^{N-1}, \quad \forall \rho \in \mathcal{A}. \tag{3.7}$$

That is, $\tilde{\rho}$ minimizes the linear functional $l(\rho) = \int_{\partial D} \rho \tilde{u} \, d\mathcal{H}^{N-1}$, relative to $\rho \in \mathcal{A}$. A well established result in rearrangement theory [4, 5] ensures that $\tilde{\rho} \in \mathrm{ext}(\mathcal{A})$. This means that $\tilde{\rho} \in \mathcal{S}$, provided that the level sets of the trace of $\tilde{u}$ on ∂D are insignificant in the following sense:

$$\forall c \in \mathbb{R} : \quad \mathcal{H}^{N-1}\left(\{x \in \partial D \mid \tilde{u}(x) = c\}\right) = 0, \tag{3.8}$$

Therefore, (3.8) provides an accessible criterion to verify whether (1.5) is solvable or not. This can also be very useful for numerical simulations, if applicable.

4 Approximation scheme

The following result provides an affirmative answer to Question 1.1:

Theorem 4.1. *Assume that* $1 < p < \infty$, $f_0 \in L^p(D)$, $\{f_n\}_{n\in\mathbb{N}} \subseteq L^p(D)$, *and let* Φ *be a functional on* $L^p(D)$, *all satisfying the following conditions:*

(i) $f_n \to f_0$ *in* $L^p(D)$.

(ii) Φ *is strictly convex and weakly continuous on* $\overline{co}\left(\cup_{n=0}^{\infty} \mathcal{R}(f_n)\right)$.

(iii) There exists a unique $\hat{f}_n \in \mathcal{R}(f_n)$ *such that*

$$\forall n \in \mathbb{N} : \; \Phi(\hat{f}_n) = \inf_{f \in \mathcal{R}(f_n)} \Phi(f) = \inf_{f \in \overline{\mathcal{R}(f_n)}^w} \Phi(f).$$

Then, $\hat{f}_n \to \hat{f}_0$ *in* $L^p(D)$.

Remark 4.1. From Lemma 2.5 (ii), we know that:

$$\bigcup_{n=0}^{\infty} \overline{\mathcal{R}(f_n)}^w \subseteq \overline{\mathrm{co}}\left(\bigcup_{n=0}^{\infty} \overline{\mathcal{R}(f_n)}^w\right) = \overline{\mathrm{co}}\left(\bigcup_{n=0}^{\infty} \mathcal{R}(f_n)\right). \tag{4.1}$$

In many cases, one can prove that Φ is strictly convex and weakly continuous on a larger set $F \supseteq \overline{\mathrm{co}}\left(\cup_{n=0}^{\infty} \mathcal{R}(f_n)\right)$ (see e. g. [12, Lemma 3.3] and [16, Lemma 3.1]). For condition (iii) in Theorem 4.1, the uniqueness of the minimizer is ensured as a consequence of Φ being strictly convex and each $\overline{\mathcal{R}(f_n)}^w$ being a convex set.

The common method of proving the existence and uniqueness of (1.7) is to first relax the problem by extending the rearrangement class to its weak closure. In these cases the second equality in (4.1) holds (see [12, 16] for details).

Finally, if the existence and uniqueness of solution are ensured for the corresponding maximization problem,[6] then one can also replace 'inf' by 'sup' in condition (iii).

[6] which is often the case for radial domains, e. g. [9, Theorem 3.5].

We break the proof of Theorem 4.1 into several lemmas.

Lemma 4.2. *Let E be a bounded subset of $L^p(D)$, and Φ be a weakly continuous functional on $\overline{E}^w$. Then, Φ is uniformly continuous on $\overline{E}^w$.*

Proof. We argue by contradiction. Suppose that Φ is not uniformly continuous on $\overline{E}^w$ and that there exists an $\epsilon > 0$ such that

$$\forall\, n \in \mathbb{N},\ \exists\, x_n, y_n \in \overline{E}^w : \|x_n - y_n\|_p < \frac{1}{n} \text{ and } |\Phi(x_n) - \Phi(y_n)| \geq \epsilon. \tag{4.2}$$

So, we have $x_n - y_n \to 0$ in $L^p(D)$. Since E is bounded, $\overline{E}^w$ is also bounded in $L^p(D)$. Hence, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that $x_{n_k} \rightharpoonup \hat{x}$ and $y_{n_k} \rightharpoonup \hat{y}$, and it should be clear that $\hat{x}, \hat{y} \in \overline{E}^w$. By using the weak continuity of Φ and (4.2), we have

$$|\Phi(\hat{x}) - \Phi(\hat{y})| \geq \epsilon > 0. \tag{4.3}$$

But then $x_n - y_n \to 0$ implies that $\hat{x} = \hat{y}$, which contradicts (4.3). □

Lemma 4.3. *Let $f, g \in L^p(D)$ and $\tilde{f} \in \mathcal{R}(f)$. Then, there exists a $\tilde{g} \in \mathcal{R}(g)$ which satisfies:*

$$\left\|\tilde{g} - \tilde{f}\right\|_p = \left\|g^\Delta - f^\Delta\right\|_p \leq \|g - f\|_p. \tag{4.4}$$

Proof. By using Lemma 2.6 (i) we can prove the existence of a measure preserving map $\rho : D \to (0, |D|)$ such that $\tilde{f} = f^\Delta \circ \rho$. Then we define $\tilde{g} := g^\Delta \circ \rho$, and we will have $\tilde{g} \in \mathcal{R}(g)$ and $\left\|\tilde{g} - \tilde{f}\right\|_p = \left\|g^\Delta - f^\Delta\right\|_p$. By applying Lemma 2.6 (ii), the assertion follows. □

Lemma 4.4. *Let f_0 and Φ be as in Theorem 4.1. For $\alpha > 0$ and $h \in L^p(D)$, we define:*

$$A(\alpha, h) = \{g \in \mathcal{R}(f_0) : \|g - h\|_p \geq \alpha\},$$

and

$$\gamma(\alpha, h) = \begin{cases} \inf_{g \in A(\alpha,h)} \Phi(g) - \Phi(h) & \text{if } A(\alpha, h) \neq \emptyset \\ \infty & \text{if } A(\alpha, h) = \emptyset \end{cases} \tag{4.5}$$

If $A(\alpha, \hat{f}_0)$ is not empty then $\gamma(\alpha, \hat{f}_0)$ is positive.

Proof. Let $\{g_n\} \subseteq A(\alpha, \hat{f}_0)$ be a minimizing sequence such that

$$\Phi(g_n) \leq \inf_{g \in A(\alpha, \hat{f}_0)} \Phi(g) + \frac{1}{n}.$$

By Lemma 2.5 (i), we have $\|g_n\|_p = \|f_0\|_p$ for every $n \in \mathbb{N}$. Hence, there exists a subsequence, still denoted $\{g_n\}$, such that $g_n \rightharpoonup \bar{g}$ in $L^p(D)$. Observe that $\bar{g} \in \overline{\mathcal{R}(f_0)}^w$. By weak continuity of Φ, we deduce:

$$\Phi(\bar{g}) \leq \inf_{g \in A(\alpha, \hat{f}_0)} \Phi(g).$$

Then, we claim $\bar{g} \neq \hat{f}_0$, which we prove by contradiction. So, let us assume that $\bar{g} = \hat{f}_0$. By condition (iii) in Theorem 4.1 we have $\hat{f}_0 \in \mathcal{R}(f_0)$. We can now apply Theorem 2.7 to deduce that $g_n \to \hat{f}_0$ in $L^p(D)$, which contradicts the definition of $A(\alpha, \hat{f}_0)$.

As a result, again by condition (iii) it follows that

$$\Phi(\hat{f}_0) < \Phi(\bar{g}) \leq \inf_{g \in A(\alpha, \hat{f}_0)} \Phi(g).$$

This completes the proof of the lemma. □

In fact, we can use rearrangement theory instead of Theorem 2.7 to prove Lemma 4.4. The following alternative proof is presented to showcase the power of rearrangement theory.

Alternative proof of Lemma 4.4: We begin the proof with two observations. First, from condition (iii) in Theorem 4.1, $\hat{f}_0$ is the unique minimizer of Φ relative to $\overline{\mathcal{R}(f_0)}^w$. Second, $\gamma(\alpha, \hat{f}_0)$ is already non-negative, so to finish the proof of the lemma we only need to rule out the possibility of $\gamma(\alpha, \hat{f}_0)$ being zero.

For simplicity, we set $A \equiv A(\alpha, \hat{f}_0)$. Then, note that A^c, the complement of A relative to $\mathcal{R}(f_0)$, is equal to the set $\{g \in \mathcal{R}(f_0) : \|g - \hat{f}_0\|_p < \alpha\}$; which is a strongly open subset of $\mathcal{R}(f_0)$. By Lemma 2.5 (iii), there exists a weakly open set W such that $\hat{f}_0 \in W \subseteq A^c$. Without loss of generality, we can choose $W = \{g \in \mathcal{R}(f_0) : |\ell(g) - \ell(\hat{f}_0)| < \epsilon\}$, for some $\epsilon > 0$ and $\ell \in (L^p)^* = L^{p'}$. Since $A \subseteq W^c$, clearly $\inf_A \Phi(g) \geq \inf_{W^c} \Phi(g)$. Hence, it suffices to show that $\inf_{W^c} \Phi(g) > \Phi(\hat{f}_0)$.

To seek a contradiction we assume that $\inf_{W^c} \Phi(g) = \Phi(\hat{f}_0)$, and let $\{g_n\} \subseteq W^c$ be a minimizing sequence. After passing to a subsequence, if necessary, and still denoted $\{g_n\}$, we infer $g_n \rightharpoonup \bar{g}$, for some $\bar{g} \in \overline{\mathcal{R}(f_0)}^w$. Since $\ell(g_n) \to \ell(\bar{g})$, we have $\bar{g} \in E \equiv \{g \in \overline{\mathcal{R}(f_0)}^w : |\ell(g) - \ell(\hat{f}_0)| \geq \epsilon\}$. On the other hand, by the weak continuity of Φ, we get $\Phi(g_n) \to \Phi(\bar{g})$.

So we must have $\Phi(\bar{g}) = \Phi(\hat{f}_0)$. Since Φ is strictly convex, then $\bar{g} = \hat{f}_0$. Whence, $\hat{f}_0 \in E$, which is a contradiction. □

Proof of Theorem 4.1: In order to derive a contradiction, we assume that there exists an $\epsilon > 0$ and a subsequence of $\{f_n\}$, still denoted $\{f_n\}$, such that $\left\|\hat{f}_n - \hat{f}_0\right\|_p \geq \epsilon$ for all $n \in \mathbb{N}$. Then, by Lemma 4.3, for all n there exist $g_n \in \mathcal{R}(f_0)$ and $h_n \in \mathcal{R}(f_n)$ such that

$$\begin{cases} \left\|\hat{f}_n - g_n\right\|_p = \left\|f_n^\Delta - f_0^\Delta\right\|_p \leq \|f_n - f_0\|_p\,, \\ \left\|h_n - \hat{f}_0\right\|_p = \left\|f_n^\Delta - f_0^\Delta\right\|_p \leq \|f_n - f_0\|_p\,. \end{cases} \tag{4.6}$$

Since $f_n \to f_0$ in $L^p(D)$, there exists $N_1 \in \mathbb{N}$ such that $\left\|\hat{f}_n - g_n\right\|_p \leq \frac{\epsilon}{2}$ for all $n \geq N_1$. As we have assumed $\left\|\hat{f}_n - \hat{f}_0\right\|_p \geq \epsilon$, we have

$$\left\|g_n - \hat{f}_0\right\|_p \geq \left\|\hat{f}_n - \hat{f}_0\right\|_p - \left\|\hat{f}_n - g_n\right\|_p \geq \frac{\epsilon}{2}, \quad \forall\, n \geq N_1. \tag{4.7}$$

Let us now refer back to Lemma 4.4. As $\{g_n\} \subseteq \mathcal{R}(f_0)$, we have $A(\frac{\epsilon}{2}, \hat{f}_0) \neq \emptyset$, which implies that $0 < \gamma(\frac{\epsilon}{2}, \hat{f}_0) < \infty$. Then, from condition (i), (4.6) and Lemma 4.2, we infer the existence of $N_2 \geq N_1$ such that

$$\forall n \geq N_2 : \begin{cases} \left|\Phi(\hat{f}_n) - \Phi(g_n)\right| < \frac{1}{2}\gamma\left(\frac{\epsilon}{2}, \hat{f}_0\right) \\ \left|\Phi(h_n) - \Phi(\hat{f}_0)\right| < \frac{1}{2}\gamma\left(\frac{\epsilon}{2}, \hat{f}_0\right). \end{cases} \tag{4.8}$$

Therefore, by using (4.7), (4.8) and Lemma 4.4, we get

$$\Phi(\hat{f}_n) > \Phi(g_n) - \frac{1}{2}\gamma\left(\frac{\epsilon}{2}, \hat{f}_0\right) \geq \Phi(\hat{f}_0) + \frac{1}{2}\gamma\left(\frac{\epsilon}{2}, \hat{f}_0\right) > \Phi(h_n), \quad \forall\, n \geq N_2.$$

which is a contradiction as $h_n \in \mathcal{R}(f_n)$. This completes the proof. □

Remark 4.2. By analyzing the proof, it can be seen that the condition (i) of Theorem 4.1 may be relaxed to $f_n^\Delta \to f_0^\Delta$ in $L^p(0, |D|)$.

Question 1.2 is answered in the following theorem:

Theorem 4.5. *Let $p \in (1, \infty)$ and assume that $f_n \to f$ in $L^p(D)$. Then:*

$$\lim_{n\to\infty} d_H(\overline{\mathcal{R}_n}^w, \overline{\mathcal{R}}^w) = 0.$$

Here, $\overline{\mathcal{R}_n}^w$ and $\overline{\mathcal{R}}^w$ are the weak closures of $\mathcal{R}_n \equiv \mathcal{R}(f_n)$ and $\mathcal{R} \equiv \mathcal{R}(f)$ in $L^p(D)$, respectively.

Proof. Let $\xi_n \in \mathcal{R}_n$. By Lemma 2.6 (i), $\xi_n = \xi_n^\Delta \circ \rho_n$, for some measure preserving map $\rho_n : D \to (0, |D|)$. Thus:

$$\|\xi_n^\Delta \circ \rho_n - f^\Delta \circ \rho_n\|_p = \|\xi_n^\Delta - f^\Delta\|_p = \|f_n^\Delta - f^\Delta\|_p \leq \|f_n - f\|_p, \tag{4.9}$$

where the inequality in (4.9) follows from Lemma 2.6 (ii).

Let us now fix an $\epsilon > 0$. Since $f_n \to f$ in $L^p(D)$, we infer the existence of $N \in \mathbb{N}$ such that:

$$\|\xi_n - f^\Delta \circ \rho_n\|_p \leq \|f_n - f\|_p < \epsilon, \quad \forall n \geq N, \tag{4.10}$$

where we have used (4.9) and the fact that $\xi_n = \xi_n^\Delta \circ \rho_n$. Note that $f^\Delta \circ \rho_n \in \mathcal{R}$. Hence, from (4.10) we deduce that $\xi_n \in \mathcal{R} + B_\epsilon(0)$, in which $B_\epsilon(0) = \{h \in L^p(D) \mid \|h\|_p < \epsilon\}$. In particular we obtain $\xi_n \in \overline{\mathcal{R}}^w + B_\epsilon(0)$ for all $n \geq N$. Thus, $\mathcal{R}_n \subseteq \overline{\mathcal{R}}^w + B_\epsilon(0)$ for all $n \geq N$.

Let us now fix $n \geq N$, and consider $\eta \in \overline{\mathcal{R}_n}^w$. Then, there exists a sequence $\{\eta_i\} \subseteq \mathcal{R}_n$ such that $\eta_i \rightharpoonup \eta$ in $L^p(D)$. Note that $\eta_i \in \overline{\mathcal{R}}^w + B_\epsilon(0)$ for all $i \in \mathbb{N}$. Therefore there exists $g_i \in \overline{\mathcal{R}}^w$ such that $\|\eta_i - g_i\|_p < \epsilon$. Since $\{g_i\}$ is bounded in $L^p(D)$, we can pass to a subsequence if necessary, still denoted $\{g_i\}$, such that $g_i \rightharpoonup g$ in $L^p(D)$. This in turn implies that $g \in \overline{\mathcal{R}}^w$.

Furthermore, we have $\eta_i - g_i \rightharpoonup \eta - g$ in $L^p(D)$. Thus, from the weak lower semicontinuity of the L^p-norm we obtain $\|\eta - g\|_p \leq \liminf_{i\to\infty} \|\eta_i - g_i\|_p < 2\epsilon$. Whence, $\eta \in g + B_{2\epsilon}(0) \subseteq \overline{\mathcal{R}}^w + B_{2\epsilon}(0)$. This shows that:

$$\overline{\mathcal{R}_n}^w \subseteq \overline{\mathcal{R}}^w + B_{2\epsilon}(0), \quad \forall n \geq N. \tag{4.11}$$

Similarly, one can prove that

$$\overline{\mathcal{R}}^w \subseteq \overline{\mathcal{R}_n}^w + B_{2\epsilon}(0), \quad \forall n \geq N. \tag{4.12}$$

From (4.11), (4.12) and Definition 2.5, we find that $d_H(\overline{\mathcal{R}_n}^w, \overline{\mathcal{R}}^w) < 2\epsilon$ for all $n \geq N$. This completes the proof of the theorem. □

5 A note on the computational aspects

The results of this paper, though presented in a theoretical framework, have implications for the study of computational aspects of the rearrangement optimization problems in general, and the two problems (1.4) and (1.5) in particular.

Whether one chooses to study rearrangement optimization problems in a more theoretical setting such as Type-2 Theory of Effectivity (TTE) [21], or the more practical framework of numerical analysis, the results of Section 4 guarantee the soundness of any appropriate discretization scheme.

We have provided enough ingredients for implementing an algorithm for the maximization problem (1.4). This is mainly as a result of Theorem 3.1, which guarantees that an answer can be found in the rearrangement class $\mathcal{S}$ defined in (1.3). As the generator is a characteristic function, the search space is more manageable. In fact, in a discretized setting, every element of $\mathcal{S}$ can (essentially) be represented as a simple array of zeros and ones.

This is not the case with minimization. As Theorem 3.6 shows, in general the unique solution may fall outside $\mathcal{S}$. Of course one could always devise a numerical method to search the larger function space $\mathcal{A}$ as defined in (2.4), but the efficiency may not be as good as the one on the smaller set $\mathcal{S}$. On the other hand, by Remark 3.2 in the special cases where condition (3.8) is satisfied on the level sets of the trace of $\tilde{u}$ on ∂D, the minimizer also falls inside $\mathcal{S}$.

A numerical implementation of the gradient method for the optimization problems (1.4) and (1.5) can be written using the formula for Gâteaux derivative as provided by Lemma 2.4 (iii). Of course we have not ruled out the possibility of local (non-global) optima. This means that a simple gradient search may get stuck in a local optimum. One way of dealing with this problem is to inject some randomness into the algorithm so that it gets a chance to escape local optima in order to reach a global optimum. Simulated annealing [15] is an example of such a randomized method.

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