Towards a Quantitative Theory of Concurrent Probabilistic Systems

Franck van Breugel

(joint work with Qiyi Tang)

August 27, 2016

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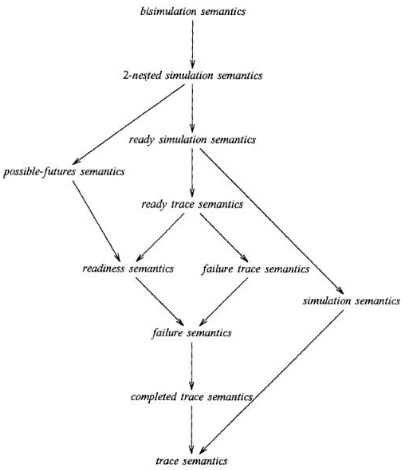
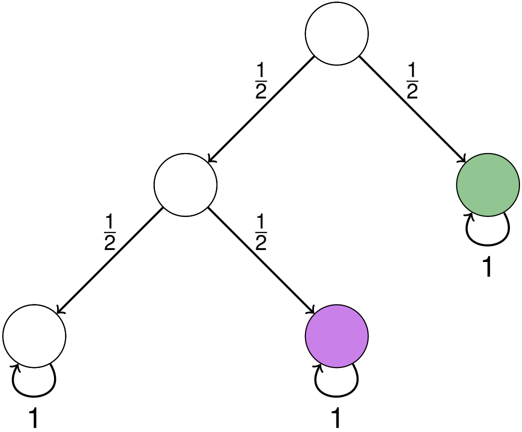


Rob van Glabbeek presented his linear time-branching time spectrum at the first CONCUR.

Rob van Glabbeek. The Linear Time-Branching Time Spectrum. CONCUR 1990.

Model of Probabilistic System



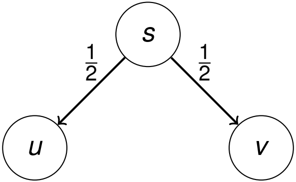
Labelled Markov chain

Andrey Markov produced the first results for Markov chains in 1906.

Transitions

$$\tau \in S \to Dist(S)$$

For each state s, the transitions of s are presented by a probability distribution $\tau(s)$ on S.



$$\tau(s)(w) = \begin{cases} \frac{1}{2} & \text{if } w = u \\ \frac{1}{2} & \text{if } w = v \\ 0 & \text{otherwise} \end{cases}$$

Definition

An equivalence relation $\mathcal{R}$ is a *probabilistic bisimulation* if for all $(s, t) \in R$,

- $\ell(s) = \ell(t)$ and
- $(\tau(s), \tau(t)) \in \bar{R}$.

Definition

Probabilistic bisimilarity is the largest probabilistic bisimulation.

Kim Larsen and Arne Skou introduced probabilistic bisimilarity in 1989.

Lifting

Definition

Let $R \subseteq S \times S$ be an equivalence relation. The *lifting* of R, $\bar{R} \subseteq Dist(S) \times Dist(S)$, is defined by

$$(\mu, \nu) \in \bar{R} \text{ if } \mu([s]) = \nu([s]) \text{ for all } s \in S$$

Next, we will provide an alternative characterization of lifting.

Coupling

Definition

A *coupling* of probability distributions μ and ν on S is a probability distribution ω on $S \times S$ with marginals μ and ν, that is, for all u, $v \in S$,

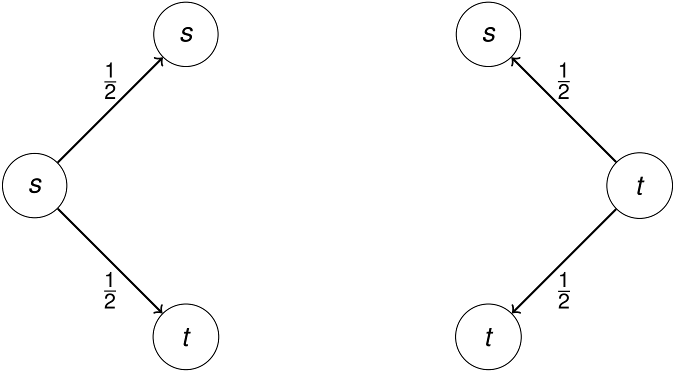
$$\sum_{v \in S} \omega(u, v) = \mu(u)$$

$$\sum_{u \in S} \omega(u, v) = \nu(v)$$

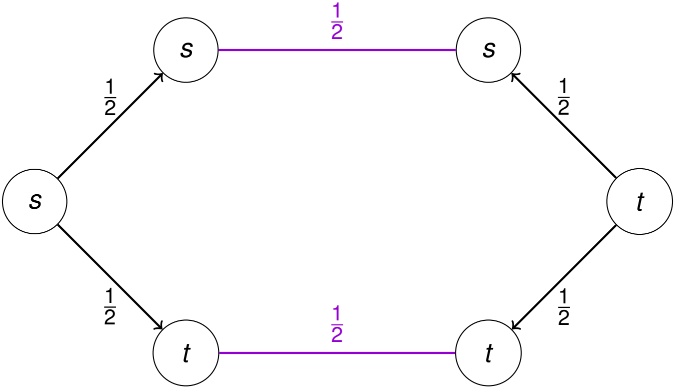
Wolfgang Doeblin introduced the notion of a coupling in 1936 (published in 1938).

The set of couplings of μ and ν is denoted by $\Omega(\mu, \nu)$.

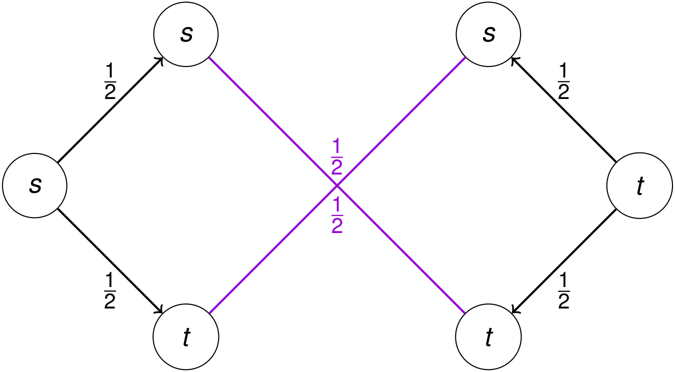
Coupling



Coupling



Coupling



Alternative characterization of lifting

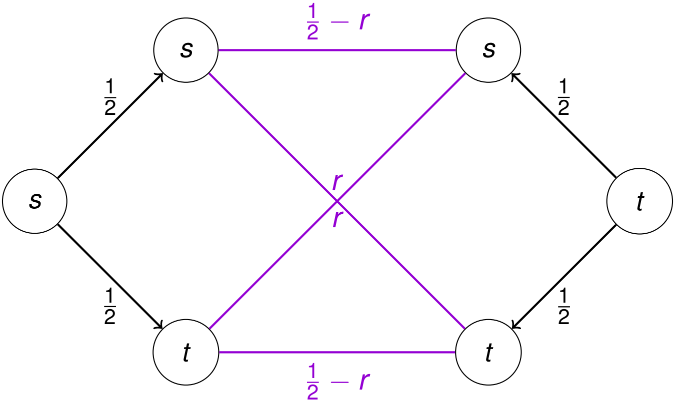
Theorem

Let $R \subseteq S \times S$ be an equivalence relation.

$$(\mu, \nu) \in \bar{R} \text{ iff } \exists \omega \in \Omega(\mu, \nu) : \text{support}(\omega) \subseteq R$$

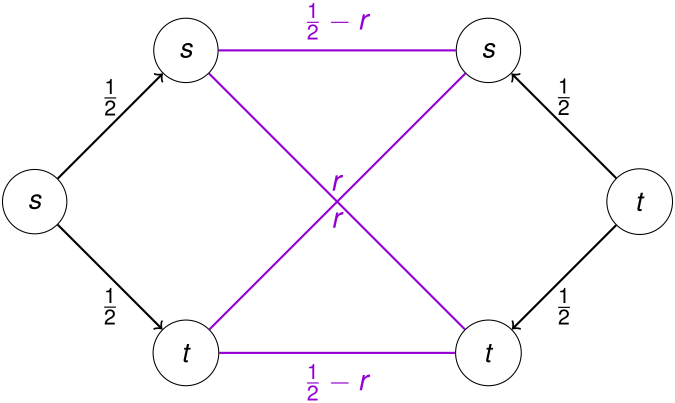
Bengt Jonsson and Kim Larsen provided the alternative characterization in 1991.

There are infinitely many couplings ($r \in [0, \frac{1}{2}]$).



Coupling

The couplings form a convex polytope, which has finitely many vertices ($r \in \{0, \frac{1}{2}\}$).



Alternative characterization of lifting

Theorem (TB 2016)

Let $R \subseteq S \times S$ be an equivalence relation.

$$(\mu, \nu) \in \bar{R} \text{ iff } \exists \omega \in V(\Omega(\mu, \nu)) : \mathrm{support}(\omega) \subseteq R$$

Alternative characterization of lifting

Theorem (TB 2016)

Let $R \subseteq S \times S$ be an equivalence relation.

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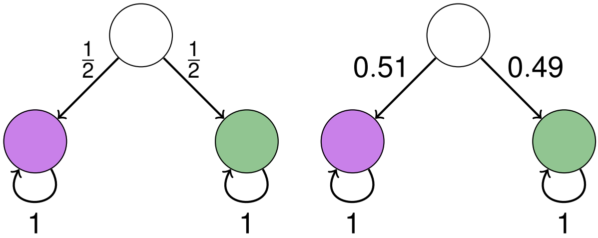
Proof sketch

- Order the states $s_1, \ldots, s_n$ such that equivalent states are consecutive.
- Apply the North-West corner method.
- Prove some loop invariants.

Fundamental problem

Behavioural equivalences are not robust for systems with real-valued data.



Scott Smolka observed that probabilistic bisimilarity, the most well-known behavioural equivalence for probabilistic systems, is not robust in 1990.

Behavioural pseudometric

Fundamental problem

Behavioural equivalences are not robust for systems with real-valued data.

Robust alternative

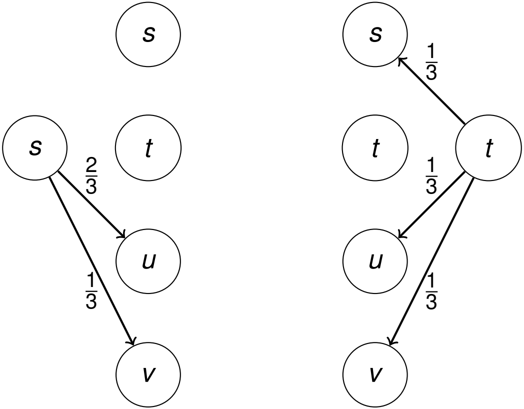
Instead of an equivalence relation

$$\sim : S \times S \to \{\text{true}, \text{false}\}$$

use a pseudometric

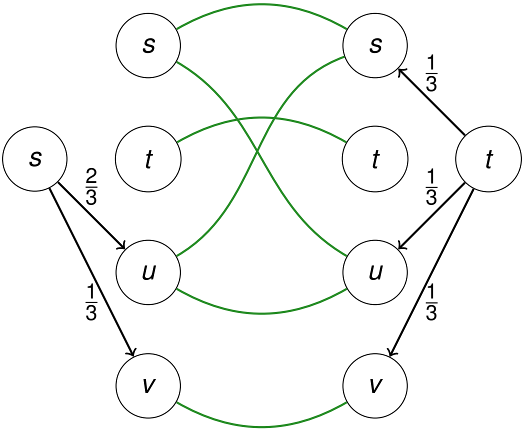
$$d : S \times S \to [0, 1].$$

Probabilistic bisimilarity



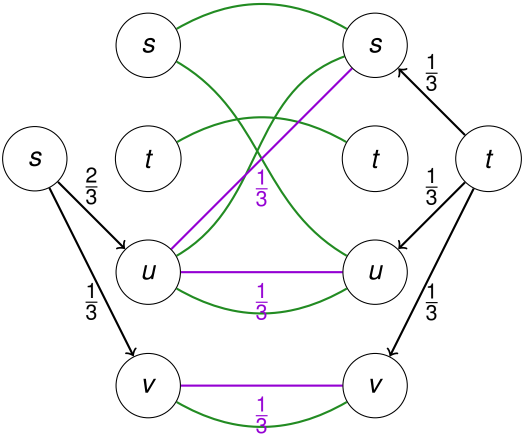
$$\text{support}(\omega) \subseteq R$$

Probabilistic bisimilarity



$$\text{support}(\omega) \subseteq R$$

Probabilistic bisimilarity



$$\text{support}(\omega) \subseteq R$$

Probabilistic bisimilarity

Let us represent the equivalence relation R with the following distance function.

$$r(s,t) = \begin{cases} 0 & \text{if } (s,t) \in R \\ 1 & \text{otherwise} \end{cases}$$

Probabilistic bisimilarity

Let us represent the equivalence relation R with the following distance function.

$$r(s,t) = \begin{cases} 0 & \text{if } (s,t) \in R \\ 1 & \text{otherwise} \end{cases}$$

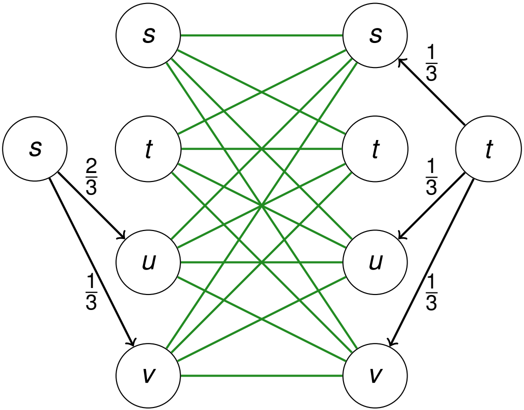
Then the condition

$$\text{support}(\omega) \subseteq R$$

is equivalent to

$$\sum_{u,v \in S} \omega(u,v)\, r(u,v) = 0$$

Quantitative generalization of probabilistic bisimilarity



$$\text{minimize} \sum_{u,v \in S} \omega(u, v)\, d(u, v)$$

Quantitative generalization of probabilistic bisimilarity

Definition

Probabilistic bisimilarity is the largest equivalence relation $\sim$ such that $s \sim t$ implies

- $\ell(s) = \ell(t)$ and
- $\exists \omega \in V(\Omega(\tau(s), \tau(t))) : \mathrm{support}(\omega) \subseteq R$.

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Definition

The *probabilistic bisimilarity pseudometric* is the smallest $d : S \times S \to [0, 1]$ such that

$$d(s,t) = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(t) \\ \displaystyle\min_{\omega \in V(\Omega(\tau(s),\tau(t)))} \sum_{u,v \in S} \omega(u,v)\, d(u,v) & \text{otherwise} \end{cases}$$

Probabilistic bisimilarity pseudometric

Josee Desharnais, Vineet Gupta, Radha Jagadeesan and Prakash Panangaden. Metrics for Labeled Markov Systems. CONCUR 1999.



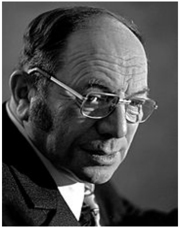

Theorem (DGJP 1999)

$s \sim t$ if and only if $d(s, t) = 0$.

Let $\mu, \nu \in Dist(S)$ and $d : S \times S \to [0, 1]$.

$$\max_{f \in (S,d) \nrightarrow [0,1]} \sum_{s \in S} f(s)\ (\mu(s) - \nu(s))$$

$$= \min_{\omega \in \Omega(\mu,\nu)} \sum_{u.v \in S} \omega(u, v)\ d(u, v)$$



Leonid Kantorovich first published this metric in 1942.

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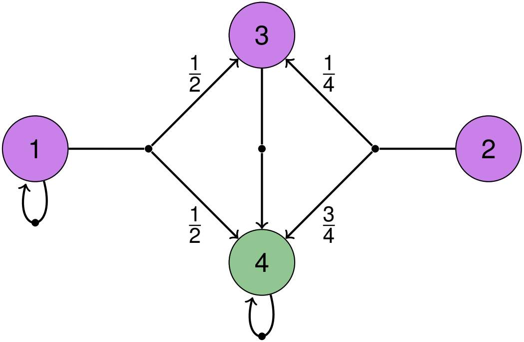
Model of Concurrent Probabilistic System

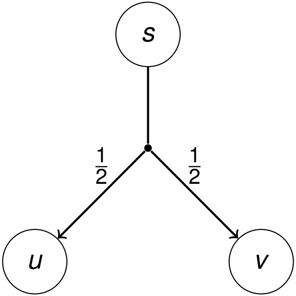


Probabilistic automaton

Roberto Segala studied probabilistic automata in his PhD thesis.

$$\rightarrow \subseteq S \times Dist(S)$$

Instead of $(s, \mu) \in \rightarrow$, we write $s \rightarrow \mu$.



$$\mu(w) = \begin{cases} \frac{1}{2} & \text{if } w = u \\ \frac{1}{2} & \text{if } w = v \\ 0 & \text{otherwise} \end{cases}$$

Probabilistic bisimulation

Definition

An equivalence relation $\mathcal{R}$ is a *probabilistic bisimulation* if for all $(s, t) \in R$,

- $\ell(s) = \ell(t)$ and
- for all $s \to \mu$ there exists $t \to \nu$ such that $(\mu, \nu) \in \bar{R}$.

Roberto Segala and Nancy Lynch. Probabilistic Simulations for Probabilistic Processes. CONCUR 1994.

Labelled Markov chain

Definition

Probabilistic bisimilarity is the largest equivalence relation $\sim$ such that $s \sim t$ implies

- $\ell(s) = \ell(t)$ and
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Definition

The *probabilistic bisimilarity pseudometric* is the smallest $d : S \times S \to [0, 1]$ such that

$$d(s, t) = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(t) \\ \min\limits_{\omega \in V(\Omega(\tau(s), \tau(t)))} \sum\limits_{u,v \in S} \omega(u, v)\, d(u, v) & \text{otherwise} \end{cases}$$

Probabilistic automaton

Definition

Probabilistic bisimilarity is the largest equivalence relation $\sim$ such that $s \sim t$ implies

- $\ell(s) = \ell(t)$ and
- $\forall s \to \mu : \exists t \to \nu : \exists \omega \in V(\Omega(\mu, \nu)) : \text{support}(\omega) \subseteq R$.

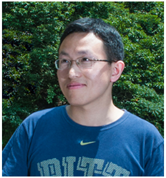
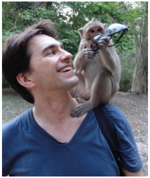
Definition

The *probabilistic bisimilarity pseudometric* is the smallest $d : S \times S \to [0, 1]$ such that if $\ell(s) \neq \ell(t)$

$$d(s,t) = \begin{cases} \max \left\{ \max_{s \to \mu} \min_{t \to \nu} \min_{\omega \in V(\Omega(\mu,\nu))} \sum_{u,v \in S} \omega(u,v)\, d(u,v), \right. \\ \left. \max_{t \to \nu} \min_{s \to \mu} \min_{\omega \in V(\Omega(\mu,\nu))} \sum_{u,v \in S} \omega(u,v)\, d(u,v) \right\} \end{cases}$$

Probabilistic bisimilarity pseudometric

Yuxin Deng, Tom Chothia, Catuscia Palamidessi and Jun Pang.
Metrics for Action-labelled Quantitative Transition Systems.
QAPL 2005.

Theorem (DCPP 2005)

$s \sim t$ if and only if $d(s, t) = 0$.

Hausdorff metric

Let $A, B \subseteq S$ and $d : S \times S \to [0, 1]$.

$$\max \left\{ \max_{s \in A} \min_{t \in B} d(s, t), \max_{t \in B} \min_{s \in A} d(s, t) \right\}$$

Felix Hausdorff introduced this metric in 1914.

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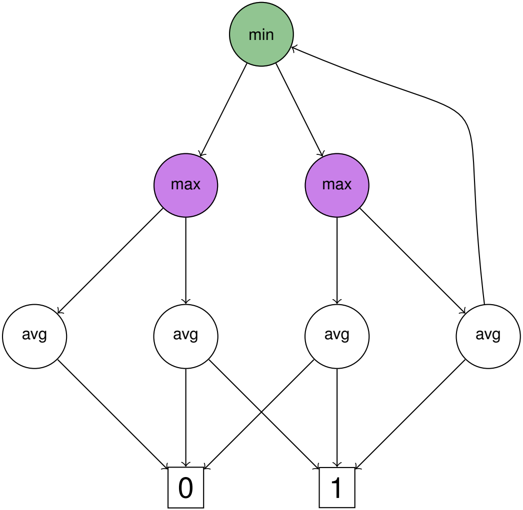
Question

How to compute the probabilistic bisimilarity distances for a probabilistic automaton?

Simple stochastic game (SSG)

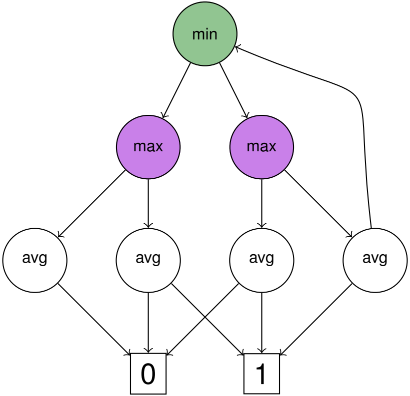


Anne Condon was the first to study simple stochastic games from a computational point of view in 1992.

Values of a SSG

Definition

The *value* of a vertex is the probability that the max player wins the game (reaches 1) provided that both players use optimal strategies (the min player tries not to reach 1).



For each probabilistic automaton we construct a corresponding simple stochastic game such that

PA	SSG
distance	value
algorithm	simple policy iteration

Ronald Howard introduced policy iteration in 1958.

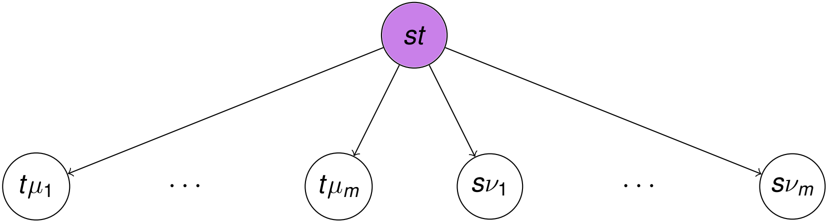
With every pair of states (s, t) of the probabilistic automaton we associate a vertex of the simple stochastic game.

If $\ell(s) \neq \ell(t)$ then $d(s, t) = 1$.

$$\boxed{1}$$

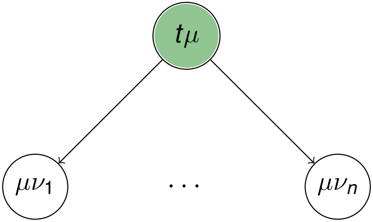
Otherwise,

$$d(s,t) = \begin{cases} \max\left\{ \max\limits_{s \to \mu} \min\limits_{t \to \nu} \min\limits_{\omega \in V(\Omega(\mu,\nu))} \sum\limits_{u,v \in S} \omega(u,v)\, d(u,v), \right. \\ \left. \max\limits_{t \to \nu} \min\limits_{s \to \mu} \min\limits_{\omega \in V(\Omega(\mu,\nu))} \sum\limits_{u,v \in S} \omega(u,v)\, d(u,v) \right\} \end{cases}$$



Otherwise,

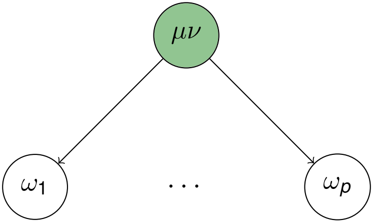
$$d(s,t) = \left\{ \begin{array}{l} \max \left\{ \max_{s \to \mu} \min_{t \to \nu} \min_{\omega \in V(\Omega(\mu,\nu))} \sum_{u,v \in S} \omega(u,v)\, d(u,v), \right. \\ \left. \qquad \max_{t \to \nu} \min_{s \to \mu} \min_{\omega \in V(\Omega(\mu,\nu))} \sum_{u,v \in S} \omega(u,v)\, d(u,v) \right\} \end{array} \right.$$



From PAs to SSGs

Otherwise,

$$d(s,t) = \begin{cases} \max \left\{ \max_{s \to \mu} \min_{t \to \nu} \min_{\omega \in V(\Omega(\mu,\nu))} \sum_{u,v \in S} \omega(u,v)\, d(u,v), \right. \\ \left. \max_{t \to \nu} \min_{s \to \mu} \min_{\omega \in V(\Omega(\mu,\nu))} \sum_{u,v \in S} \omega(u,v)\, d(u,v) \right\} \end{cases}$$



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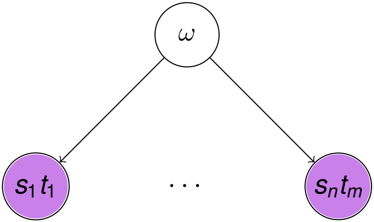

Simple policy iteration

Theorem

If

- vertices have value zero iff they are zero sinks and
- vertices have value one iff they are one sinks

then

- simple policy iteration computes a globally optimal strategy.

Distance zero and one

Theorem (DGJP 1999 and BEM 2000)

Distance zero can be decided in polynomial time.

Christel Baier, Bettina Engelen and Mila Majster-Cederbaum. Deciding Bisimilarity and Similarity for Probabilistic Processes. JCSS 2000.

Theorem (TB 2016)

Distance one can be decided in polynomial time.

From PAs to SSGs

With every pair of states (s, t) of the probabilistic automaton we associate a vertex of the simple stochastic game.

If $d(s, t) = 0$

$$\boxed{0}$$

If $d(s, t) = 1$

$$\boxed{1}$$

Trends

- Games can be used to characterize and compute behavioural pseudometrics.
- Decide the behavioural equivalence before computing the behavioural pseudometric.