System Validation: Modal μ-calculus Semantics

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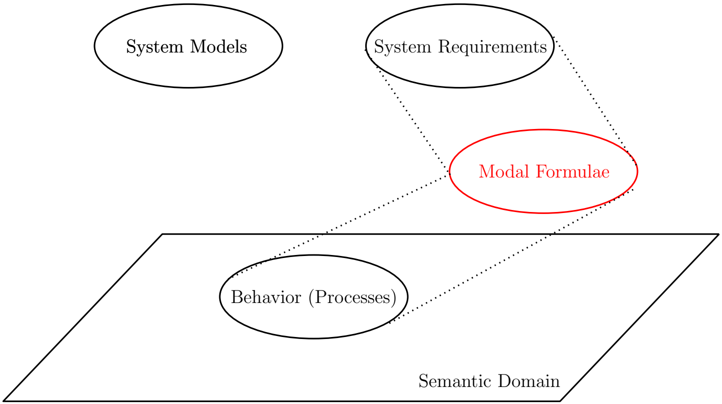
Open Universiteit

General Overview



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Some properties for which we need the μ-calculus:

- For some execution φ holds everywhere

$$\nu X.\varphi \wedge ([\mathit{true}]\mathit{false} \vee \langle \mathit{true} \rangle X)$$

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- Eventually φ will hold (in every execution)

$$\mu X.\varphi \vee (\langle \mathit{true}\rangle \mathit{true} \wedge [\mathit{true}]X)$$

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Environment

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- Use this assumption to compute solution

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Observe

- monotonic $(U \subseteq V \implies \Phi_\eta(U) \subseteq \Phi_\eta(V))$
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Therefore (Knaster-Tarski):

$$[\![\mu X.\varphi]\!]_L^\eta = \bigcup_i \Phi_\eta^i(\emptyset)$$

$$[\![\nu X.\varphi]\!]_L^\eta = \bigcap_i \Phi_\eta^i(S)$$

Computing fixed points

Due to Knaster-Tarski, use fixed point iteration

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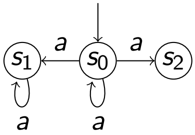
- there exists m s.t. $\Phi^{m+1}(T) = \Phi^m(T)$
- For μ, start with $T = \emptyset$, for ν start with $T = S$; iterate until m is found

Example

$$\mu X.[a]\mathit{false} \vee \langle \mathit{true} \rangle X$$



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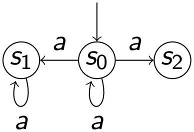

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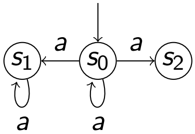

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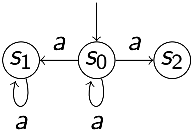

$$\Phi(T) = [\![[a]\mathit{false} \vee \langle \mathit{true} \rangle]\!]^{\eta[X:=T]}$$

$$\begin{aligned}
\Phi^0(\emptyset) &= \emptyset \\
\Phi^1(\emptyset) &= [\![[a]\mathit{false} \vee \langle \mathit{true} \rangle]\!]^{\eta[X:=\emptyset]} \\
&= [\![[a]\mathit{false}]\!]^{\eta[X:=\emptyset]} \cup [\![\langle \mathit{true} \rangle X]\!]^{\eta[X:=\emptyset]} \\
&= \{s_2\} \cup \emptyset \\
&= \{s_2\}
\end{aligned}$$

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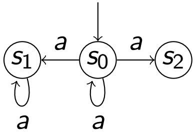

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\Phi(T) &= [\![[a]\mathit{false} \vee \langle \mathit{true}\rangle]\!]^{\eta[X:=T]} \\
\Phi^1(\emptyset) &= \{s_2\} \\
\Phi^2(\emptyset) &= [\![[a]\mathit{false} \vee \langle \mathit{true}\rangle]\!]^{\eta[X:=\{s_2\}]} \\
&= [\![[a]\mathit{false}]\!]^{\eta[X:=\{s_2\}]} \cup [\![\langle \mathit{true}\rangle X]\!]^{\eta[X:=\{s_2\}]} \\
&= \{s_2\} \cup \{s_0\} \\
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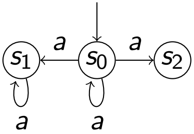

$$\Phi(T) = [\![[a]\mathit{false} \vee \langle \mathit{true} \rangle]\!]^{\eta[X:=T]}$$

$$\Phi^2(\emptyset) = \{s_0, s_2\}$$

$$\begin{aligned}
\Phi^3(\emptyset) &= [\![[a]\mathit{false} \vee \langle \mathit{true} \rangle]\!]^{\eta[X:=X^2]} \\
&= [\![[a]\mathit{false}]\!]^{\eta[X:=X^2]} \cup [\![\langle \mathit{true} \rangle X]\!]^{\eta[X:=X^2]} \\
&= \{s_2\} \cup \{s_0\} \qquad\qquad = \{s_0, s_2\} \\
&= \Phi^2(\emptyset)
\end{aligned}$$

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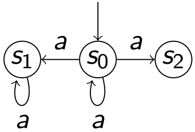

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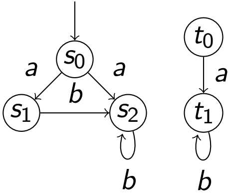
$\nu X . \langle b \rangle true \wedge [b]X$



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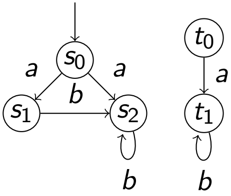

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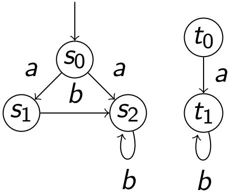

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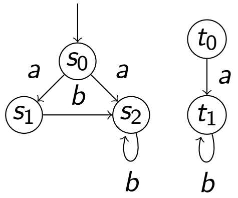

$$\Phi(T) = \langle b\rangle \mathit{true} \wedge [b]X$$

$$\begin{aligned}
\Phi^0(S) &= S \\
\Phi^1(S) &= [\![\langle b\rangle \mathit{true} \wedge [b]X]\!]^{\eta[X:=S]} \\
&= [\![\langle b\rangle \mathit{true}]\!]^{\eta[X:=S]} \cap [\![[b]X]\!]^{\eta[X:=S]} \\
&= \{s_1, s_2, t_1\} \cap \{s_0, s_1, s_2, t_0, t_1\} \\
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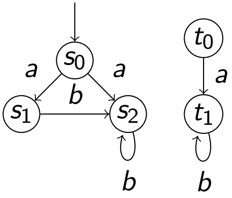

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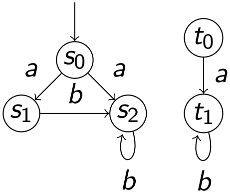

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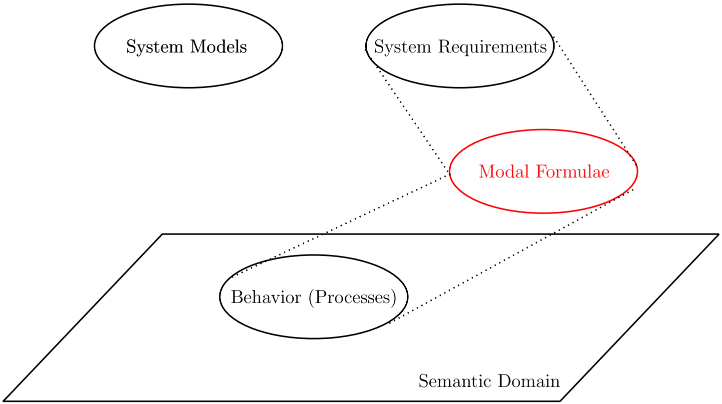

Thank you very much.