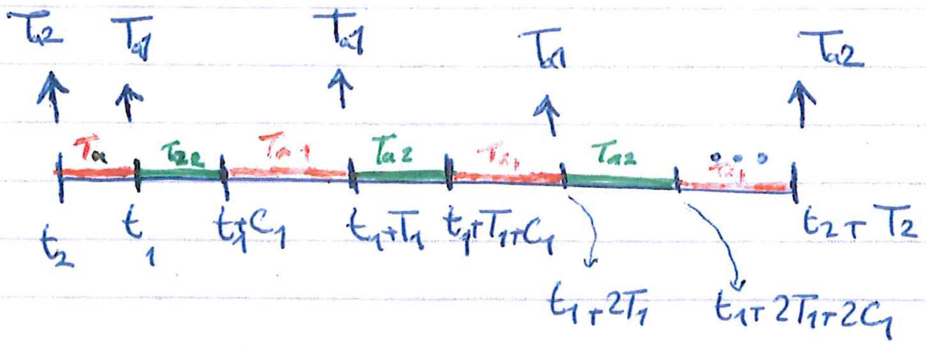


Assumptions (τ_1 and τ_2)

- Two tasks with periods $T_1 < T_2$,
- WCETs resp C_1 and C_2, and
- the scheduling regime is Rate Monotonic.

Thm. The delay in the completion of τ_2 is maximum when it arrives simultaneously with τ_1.

Prf. The delay in the completion of τ_2 may be increased with the number of occurences of τ_1 within the interval $[t_2, t_2+T_2]$. The number of occurences of τ_1 is maximized to $\lceil T_2/T_1 \rceil$ with $t_2 = t_1$. $\square$

Note. The above-given task-set is schedulable only if $C_2 + \lceil T_2/T_1 \rceil C_1 \le T_2$.

Dfn. $U = \sum_{i=1}^{n} (C_i/T_i)$ (in our case $n=2$)

Thm. In the aforementioned setting, the maximum utilization is $U = 2(\sqrt[n]{2}-1) = 2(\sqrt{2}-1)$.

Prf. In order to maximize U, assume that Ta_2 uses all its available processor time in the interval $[t_2, t_2+T_2]$. (Pictorially, this means that the whole interval is either green, or red).

We distinguish the following two cases.

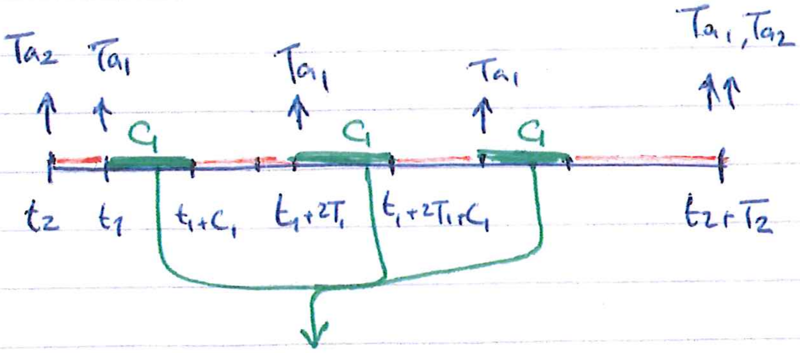
- Either all instances of Ta_1 finish within $[t_2, t_2+T_2]$, i.e.,

$$C_2 = \underbrace{T_2 - \underbrace{C_1 \overbrace{\lfloor T_2/T_1 \rfloor}^{\#\text{ inst. of } Ta_1}}_{\text{Green area}}}_{\text{Red area}} \quad (1)$$

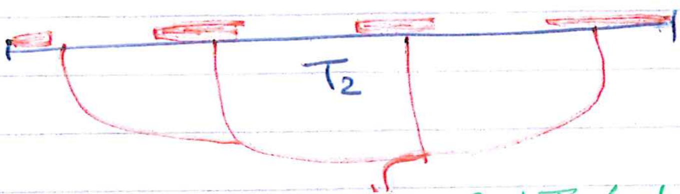
Then $U = \sum_{i=1}^{2} C_i/T_i = C_1/T_1 + C_2/T_2 \overset{(1)}{=}$

$$C_1/T_1 + \frac{T_2}{T_2} - \frac{C_1}{T_2}\lfloor T_2/T_1 \rfloor =$$

$$1 + C_1\left(1/T_1 - \frac{1}{T_2}\lfloor T_2/T_1\rfloor\right) \quad (2)$$



$C_1\lfloor T_2/T_1 \rfloor$



$T_2 - C_1\lfloor T_2/T_1 \rfloor$

- Or an occurence of Ta_1 goes beyond $t_2 + T_2$.
In this case, we have:

$$C_1 \geq T_2 - T_1 \lfloor T_2 / T_1 \rfloor \quad (3)$$

(see the boundary of $T_1 \lfloor T_2/T_1 \rfloor$ below)

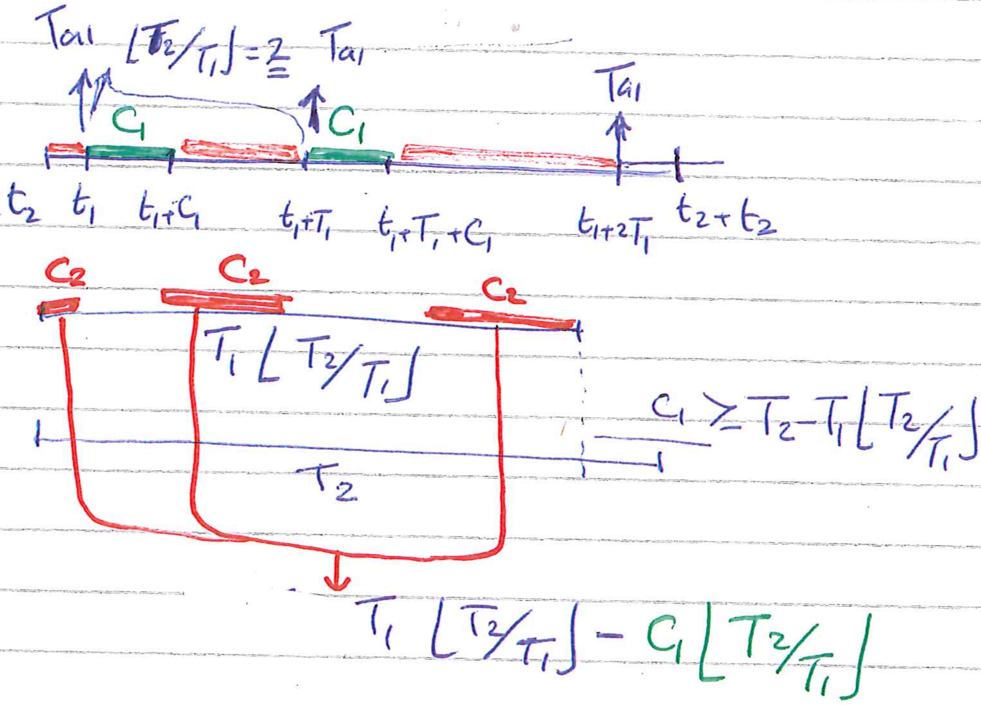
We also have:

$$C_2 = T_1 \lfloor T_2/T_1 \rfloor - C_1 \lfloor T_2/T_1 \rfloor \quad (4)$$

$$U = C_1/T_1 + C_2/T_2 \overset{(4)}{=}$$

$$C_1/T_1 + T_1/T_2 \lfloor T_2/T_1 \rfloor - C_1/T_2 \lfloor T_2/T_1 \rfloor =$$

$$T_1/T_2 \lfloor T_2/T_1 \rfloor + C_1 \left(1/T_1 - 1/T_2 \lfloor T_2/T_1 \rfloor \right) = (5)$$



The first formula for U decreases when C_1 increase, the second increases when C_1 increases; hence U is minimum at the boundary between the two, i.e., when

$$C_1 \overset{(3)}{=} T_2 - T_1 \lfloor T_2/T_1 \rfloor \text{ and}$$
$$C_2 \overset{(1)}{=} T_2 - C_1 \lfloor T_2/T_1 \rfloor \qquad (5)$$

Let $f = T_2/T_1$ and $I = \lfloor T_2/T_1 \rfloor$

$$U = C_1/T_1 + C_2/T_2 \overset{(6)}{=}$$

$$T_2/T_1 - T_1/T_1 \lfloor T_2/T_1 \rfloor + T_2/T_2 - C_1/T_2 \lfloor T_2/T_1 \rfloor \overset{??}{=}$$

$$1 - f(1-f)/I+f$$

U is minimum when I is max, i.e., $\underline{1}$.

$$U = 1 - \frac{f(1-f)}{1+f}$$