Exercise: Epipolar Geometry

1 Suggested reading

Section 13.4 of the course book (Bigun, Vision With Direction) introduces some of the terminology used in the exercise. The theory is discussed in section 13.5 and determination of the fundamental matrix in 13.6.

2 Determining the Fundamental Matrix

As you have seen during the theory classes, the projective coordinates (homogenous coordinates) of two corresponding points p_r and p_l (the notation used here is slighly different from that of the book, p represent the vector from the origin to a point in the corresponding image frame) in the image frames of the two cameras are linked by the following relation (the epipolar equation):

$$\mathsf{p}_r^T \mathsf{F} \mathsf{p}_l = 0. \tag{1}$$

Matrix F is a rank-2 3-by-3 matrix known as the *fundamental matrix*.

Determining matrix F is straightforward once you know the coordinates $\mathsf{p}_r(i)$ and $\mathsf{p}_l(i)$ for at least 8 corresponding points, since you can then obtain eight equations for the components of F (you do not need nine equations, since F is defined up to a scale factor). This way of finding F is known as the “Eight-Point Algorithm”; in practice, however, you might need more points to keep numerical errors low.

For each point i you have

$$\mathsf{p}_r(i)^T \mathsf{F} \mathsf{p}_l(i) = \sum_j \sum_k (\mathsf{p}_r(i))_j (\mathsf{p}_l(i))_k \mathsf{F}_{jk} = 0 \tag{2}$$

which you can write as the product of a row and a column vector:

$$[(\mathsf{p}_r(i))_j (\mathsf{p}_l(i))_k][\mathsf{F}_{jk}]^T = 0 \tag{3}$$

where each vector has nine components (one for all the possible values of j and k).

When you have several equations like this, you can combine them by grouping the row vectors in a matrix Q:

$$\mathsf{Q}[\mathsf{F}_{jk}]^T = 0 \tag{4}$$

where Q has one row for each pair of corresponding points. You can then easily determine $[\mathsf{F}_{jk}]^T$ by computing the eigenvalues of $\mathsf{Q}^\mathsf{T}\mathsf{Q}$ and taking the eigenvector corresponding to the smallest (in theory zero) eigenvalue.

For numerical reasons, it is better to normalize the coordinates of the points before solving the system; the next section explains how to do this.

3 Normalizing the image coordinates

It is preferable to normalize the coordinates in the image domain so that they have zero average and unit variance before proceeding with the computation above; otherwise, matrix Q can be seriously ill-conditioned.

This is done by multiplying each point p_l by matrix

$$\mathsf{H}_l = \begin{bmatrix} 1/d & 0 & -\bar{x}/d \\ 0 & 1/d & -\bar{y}/d \\ 0 & 0 & 1 \end{bmatrix} \tag{5}$$

where $\bar{x}$ is the average x component, $\bar{y}$ is the average y component and

$$d = \sqrt{\frac{\sum_i (x_i - \bar{x})^2 + \sum_i (y_i - \bar{y})^2}{2n}}, \tag{6}$$

with n equal to the number of points.

In this way, we obtain the normalized coordinates $\hat{\mathsf{p}}_l = \mathsf{H}_l \mathsf{p}_l$ and $\hat{\mathsf{p}}_r = \mathsf{H}_r \mathsf{p}_r$ that can be used in the eight-point algorithm described above. Note that H_l and H_r are not the same and they should be computed separately.

The eight point algorithm then gives us a matrix $\tilde{\tilde{\mathsf{F}}}$; this is related to the original matrix $\tilde{\mathsf{F}}$ by

$$\mathsf{F} = \mathsf{H}_r^T \tilde{\mathsf{F}} \mathsf{H}_l. \tag{7}$$

4 Finding the epipoles

The projective coordinates of the epipoles are the null spaces of matrices F and F^T. You can find them by computing the SVD

$$\mathsf{F} = \mathsf{UDV}^T \tag{8}$$

of F. The left image epipole e_l is the column of V associated with the smallest singular value; the right image epipole e_r is the column of U associated with the smallest singular value. You will remember that the smallest singular value should be zero, since the fundamental matrix F has rank two. If it is not (which is probably the case), set it to zero and recompute matrix F from the (modified) decomposition. This is the correct way of enforcing the singularity constraint, in that it entails the least possible modification of matrix F in terms of the Frobenius distance (square root of the sum of the squares of the components).

5 Verifying the epipolar constraint

Verify the epipolar constraint by drawing the lines

$$\mathsf{x}_r^T \mathsf{F} \mathsf{p}_l(i) = 0 \tag{9}$$

on the right image for at least two points $i_{1,2}$ of the left image. The corresponding points $\mathsf{p}_r(i_{1,2})$ in the right image should lie on the respective line.

Do the same in the other direction, starting from the corresponding points in the right image.

6 Documentation

Include the following items in your report:

- A discussion of the Epipolar equation and the Fundamental matrix.
- Images with the epipolar lines and the corresponding points (either use Matlab or a paint program to mark the points)

Appendix: Useful Matlab commands

eig
eigs
svd
mean
sum
imread
impixel
help